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Author(s)	Cortés, Vicente; Hasegawa, Kazuyuki
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# THE QUATERNIONIC/HYPERCOMPLEX-CORRESPONDENCE

VICENTE CORTÉS and KAZUYUKI HASEGAWA

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## Abstract

Given a quaternionic manifold  $M$  with a certain  $U(1)$ -symmetry, we construct a hypercomplex manifold  $M'$  of the same dimension. This construction generalizes the quaternionic Kähler/hyper-Kähler-correspondence. As an example of this construction, we obtain a compact homogeneous hypercomplex manifold which does not admit any hyper-Kähler structure. Therefore our construction is a proper generalization of the quaternionic Kähler/hyper-Kähler-correspondence.

## 1. Introduction

Let us recall that there exist constructions due to Andriy Haydys, called the QK/HK-correspondence and the HK/QK-correspondence, which relate quaternionic Kähler manifolds to hyper-Kähler manifolds of the same dimension [12]. These constructions have been generalized to include possibly indefinite metrics [2, 1]. In this way the supergravity c-map metric and a one-parameter deformation thereof have been described as an application of the HK/QK-correspondence with indefinite initial hyper-Kähler data. Many complete quaternionic Kähler manifolds can be obtained in this way, see for instance [8] for co-homogeneity one examples.

The main result of this paper, see Theorem 6.4, is a construction of a hypercomplex manifold from a quaternionic manifold with a  $U(1)$ -action, which we may call the *quaternionic/hypercomplex-correspondence* (Q/H-correspondence for short). This construction generalizes the QK/HK-correspondence.

In [22, 14, 21], it is shown that with every quaternionic manifold  $M$  one can associate an  $\mathbb{H}^*/\{\pm 1\}$ -bundle over  $M$  and a hypercomplex structure on the total space of the bundle. More precisely [21], there exists a one-parameter family of  $\mathbb{H}^*/\{\pm 1\}$ -bundles such that, given a quaternionic connection on  $M$ , each of the bundles is endowed with an almost hypercomplex structure. For a particular choice of the parameter, the almost hypercomplex structure is integrable and independent of the connection. Here we will adopt a different point of view. Instead of a one-parameter family of bundles, we will define a single principal  $\mathbb{H}^*/\{\pm 1\}$ -bundle, which we call the *Swann bundle*, endowed with a one-parameter family of almost hypercomplex structures (still depending on a quaternionic connection). Again we find that, for a particular choice of the parameter, namely  $c = -4(n + 1)$ , the almost hypercomplex structure is always integrable and independent of the connection, see Proposition 3.3. Here  $4n = \dim M$ . For all other values of the parameter, we show that the almost hypercomplex structure is integrable if and only if all  $I \in Q$ , where  $Q$  denotes the quaternionic structure,

are skew-symmetric with respect to the skew-symmetric part of the Ricci-curvature, see Theorem 3.6.

Now we briefly explain how we obtain the Q/H-correspondence. Given an infinitesimal automorphism  $X$  of a quaternionic manifold  $(M, Q, \nabla)$  endowed with a quaternionic connection  $\nabla$ , we show that the natural lift  $\hat{X}$  of  $X$  to the Swann bundle  $\hat{M}$  preserves each member of the one-parameter family of almost hypercomplex structures. The next step is to perform a hypercomplex reduction with respect to  $\hat{X}$ . Recall that hypercomplex reduction was introduced by Dominic Joyce in [14]. It is defined as the quotient of a level set of a moment map by the group action. The construction is based on the notion of a moment map in this context as defined in [14]. Here we define the moment map for the infinitesimal automorphism  $\hat{X}$  by the equation (5.4) and analyse Joyce's conditions in Proposition 5.8. Assuming that  $\hat{X}$  generates a free  $U(1)$ -action, we can finally perform the reduction obtaining a hypercomplex manifold  $M'$ . Otherwise, we can construct the hypercomplex structure on a submanifold transversal to the foliation defined by  $\hat{X}$  (on some open submanifold of  $\hat{M}$ ).

Note that one can find a related construction of a hypercomplex manifold from a quaternionic manifold (endowed in addition with a complex structure and a compatible  $S^1$ -action) in Proposition 5.5 in [6], in which the authors study the quaternionic Feix-Kaledin (qFK) construction. The qFK construction is a generalization of the original one in [16, 9]. In Theorem 5 of [7], it is shown that a quaternionic Kähler manifold obtained from the qFK construction and a hyper-Kähler manifold from the original Feix-Kaledin construction are related by the HK/QK-correspondence. This is analogous to the relation between the supergravity c-map and the rigid c-map [2, 1].

Examples of our Q/H-correspondence include compact homogeneous hypercomplex manifolds. Indeed, starting with a homogeneous quaternionic Hopf manifold

$$(\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\Delta_{\mathrm{U}(1)}},$$

we obtain a homogeneous hypercomplex Hopf manifold

$$(\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}$$

by the Q/H-correspondence, see Example 7.8. Note that this hypercomplex manifold does not admit any hyper-Kähler structure for topological reasons. Therefore our construction is a proper generalization of the QK/HK-correspondence.

## 2. Preliminaries

Throughout this paper, all manifolds are assumed to be smooth and without boundary and maps are assumed to be smooth unless otherwise mentioned. The space of sections of a vector bundle  $E \rightarrow M$  is denoted by  $\Gamma(E)$ .

We say that  $M$  is a *quaternionic manifold* with the quaternionic structure  $Q$  if  $Q$  is a subbundle of  $\mathrm{End}(TM)$  of rank 3 which at every point  $x \in M$  is spanned by endomorphisms  $I_1, I_2, I_3 \in \mathrm{End}(T_x M)$  satisfying

$$(2.1) \quad I_1^2 = I_2^2 = I_3^2 = -\mathrm{id}, \quad I_1 I_2 = -I_2 I_1 = I_3,$$

and there exists a torsion-free connection  $\nabla$  on  $M$  such that  $\nabla$  preserves  $Q$ , that is,  $\nabla_X \Gamma(Q) \subset$

$\Gamma(Q)$  for all  $X \in \Gamma(TM)$ . Note that we use the same letter  $\nabla$  for the connection on  $\text{End}(TM)$  induced by  $\nabla$  if there is no confusion. Such a torsion-free connection  $\nabla$  is called a *quaternionic connection* and the triplet  $(I_1, I_2, I_3)$  is called an *admissible frame* of  $Q$  at  $x$ . The dimension of a quaternionic manifold  $M$  is denoted by  $4n$ . Note that a quaternionic connection is not unique, in fact, there is the following result [10, 5].

**Lemma 2.1.** *Let  $\nabla^1$  and  $\nabla^2$  be quaternionic connections on  $(M, Q)$ . Then there exists a 1-form  $\xi$  on  $M$  such that*

$$(2.2) \quad \nabla_X^2 Y = \nabla_X^1 Y + S_X^\xi Y$$

for all  $X, Y \in \Gamma(TM)$ , where  $S^\xi$  is defined by

$$\begin{aligned} S_X^\xi Y = & \xi(X)Y + \xi(Y)X - \xi(I_1 X)I_1 Y - \xi(I_1 Y)I_1 X \\ & - \xi(I_2 X)I_2 Y - \xi(I_2 Y)I_2 X - \xi(I_3 X)I_3 Y - \xi(I_3 Y)I_3 X. \end{aligned}$$

Conversely, for a given quaternionic connection  $\nabla^1$ , the connection  $\nabla^2$  given by the equation above is also a quaternionic connection.

An *almost hypercomplex manifold* is defined to be a manifold  $M$  endowed with 3 almost complex structures  $I_1, I_2, I_3$  satisfying the quaternionic relations (2.1). If  $I_1, I_2, I_3$  are integrable, then  $M$  is called a *hypercomplex manifold*. There exists a unique torsion-free connection on a hypercomplex manifold for which the hypercomplex structures are parallel. It is called the *Obata connection* [18]. Obviously, hypercomplex manifolds are quaternionic manifolds with  $Q = \langle I_1, I_2, I_3 \rangle$ .

### 3. The canonical family of almost hypercomplex structures on the Swann bundle $\hat{M}$

In this section we will define a principal  $\mathbb{R}^{>0} \times \text{SO}(3)$ -bundle  $\hat{M} \rightarrow M$  over a quaternionic manifold  $(M, Q)$  equipped with a quaternionic connection  $\nabla$  and endow  $\hat{M}$  with a one-parameter family of almost hypercomplex structures depending on the quaternionic connection  $\nabla$ . Then we will study the integrability of the hypercomplex structure and its dependence (or independence) on the choice of  $\nabla$  for different values of the parameter.

**3.1. The principal bundle  $\hat{M} \rightarrow M$ .** Let  $S$  be the principal  $\text{SO}(3)$ -bundle of admissible frames  $(I_1, I_2, I_3)$  over a quaternionic manifold  $(M, Q)$ . The principal action  $\tau$  of  $g \in \text{SO}(3)$  is given by  $\tau(s, g) = sg^\varepsilon$  for  $s = (I_1, I_2, I_3) \in S$ , where  $\varepsilon = 1$  (resp.  $\varepsilon = -1$ ) if  $S$  is considered as a right (resp. left)-principal bundle. The bundle projection of  $S$  is denoted by  $\pi_S$ . We take a basis  $(e_1, e_2, e_3)$  of  $\mathbb{R}^3 \cong \text{Im } \mathbb{H} \cong \mathfrak{sp}(1) \cong \mathfrak{so}(3)$  so that

$$[e_\alpha, e_\beta] = 2e_\gamma$$

for any cyclic permutation  $(\alpha, \beta, \gamma)$ . Hereafter  $(\alpha, \beta, \gamma)$  will be always a cyclic permutation, whenever the three letters appear in an expression. A quaternionic connection induces a principal connection  $\theta : TS \rightarrow \mathfrak{so}(3)$  and we denote  $\theta = \sum \theta^\alpha e_\alpha$ . Moreover we consider the principal  $\mathbb{R}^{>0}$ -bundle  $S_0 := (\Lambda^{4n}(T^*M) \setminus \{0\}) / \{\pm 1\}$  over  $M$ , where  $\mathbb{R}^{>0} = \{a \in \mathbb{R} \mid a > 0\}$ . The principal  $\mathbb{R}^{>0}$ -action  $\tau_0$  on  $S_0$  is given by scalar multiplication  $\tau_0(\rho, a) := \rho a^\varepsilon$  ( $\varepsilon = \pm 1$ ) for  $\rho \in S_0$  and  $a \in \mathbb{R}^{>0}$ . The bundle projection of  $S_0$  is denoted by  $\pi_{S_0}$ . A quaternionic connection induces also a principal connection  $\theta_0 : TS_0 \rightarrow \mathbb{R} = \text{Lie}(\mathbb{R}^{>0})$ . The product  $S_0 \times S$  is a principal  $\mathbb{R}^{>0} \times \text{SO}(3)$ -bundle over  $M \times M$  whose principal action is  $\tau_0 \times \tau$ . The

$\mathbb{R}^4 (\cong \mathbb{R} \oplus \mathfrak{so}(3))$ -valued 1-form  $(\theta_0 \circ pr_{TS_0}, \theta \circ pr_{TS}) = (\theta_0 \circ pr_{TS_0}, \theta_1 \circ pr_{TS}, \theta_2 \circ pr_{TS}, \theta_3 \circ pr_{TS})$  is a principal connection on  $S_0 \times S$ , where  $pr_{TS_0}$  (resp.  $pr_{TS}$ ) is the projection from  $T(S_0 \times S) \cong TS_0 \times TS$  onto  $TS_0$  (resp.  $TS$ ). Note that the Lie group  $\mathbb{R}^{>0} \times \mathrm{SO}(3) = \mathbb{H}^*/\{\pm 1\}$ .

Let  $\Delta : M \rightarrow M \times M$  be the diagonal map defined by  $\Delta(x) = (x, x)$  for each  $x \in M$ . The pullback bundle

$$\hat{M} := \Delta^*(S_0 \times S) = \{(x, (\rho, s)) \in M \times (S_0 \times S) \mid x = \pi_{S_0}(\rho) = \pi_S(s)\}$$

is a principal  $\mathbb{R}^{>0} \times \mathrm{SO}(3)$ -bundle over  $M$  and  $\bar{\theta} := \Delta_{\#}^*(\theta_0 \circ pr_{TS_0}, \theta \circ pr_{TS})$  is a principal connection on  $\hat{M}$ , where  $\Delta_{\#} : \hat{M} \rightarrow S_0 \times S$  is the canonical bundle map. The bundle projection of  $\hat{M}$  onto  $M$  is denoted by  $\hat{\pi}$ . Using the bundle projections  $\hat{\pi}$ ,  $\pi_{S_0}$ ,  $\pi_S$  and the principal connections  $\bar{\theta}$ ,  $\theta_0$ ,  $\theta$ , we have the decomposition

$$(3.1) \quad T\hat{M} = \bar{\mathcal{V}} \oplus \bar{\mathcal{H}}, \quad TS_0 = \mathcal{V}_0 \oplus \mathcal{H}_0, \quad TS = \mathcal{V} \oplus \mathcal{H},$$

where  $\bar{\mathcal{V}} = \mathrm{Ker} \hat{\pi}_*$ ,  $\bar{\mathcal{H}} = \mathrm{Ker} \bar{\theta}$  and so on. It holds that  $(\Delta_{\#})_*(\bar{\mathcal{V}}_{(x,(\rho,s))}) = (\mathcal{V}_0)_{\rho} \times \mathcal{V}_s$  and  $(\Delta_{\#})_*(\bar{\mathcal{H}}_{(x,(\rho,s))}) \subset (\mathcal{H}_0)_{\rho} \times \mathcal{H}_s$  for each  $(x, (\rho, s)) \in \hat{M}$ . Set  $\Delta_S := pr_{TS} \circ (\Delta_{\#})_*$  and  $\Delta_{S_0} := pr_{TS_0} \circ (\Delta_{\#})_*$ . The principal actions on  $\hat{M}$ ,  $S_0 \times S$ ,  $S_0$  and  $S$  induce fundamental vector fields. We denote by  $\tilde{A}$  the fundamental vector field corresponding to a Lie algebra element  $A$ , irrespective of the manifold on which the vector field is defined, and set  $Z_{\alpha} = \tilde{e}_{\alpha}$  ( $\alpha = 1, 2, 3$ ). Note that  $[Z_{\alpha}, Z_{\beta}] = 2\varepsilon Z_{\gamma}$ .

**3.2. The canonical family of almost hypercomplex structures.** Let  $(M, Q)$  be a quaternionic manifold,  $\nabla$  a quaternionic connection and  $\hat{\pi} : \hat{M} \rightarrow M$  the principal  $\mathbb{R}^{>0} \times \mathrm{SO}(3)$ -bundle with connection  $\bar{\theta}$  constructed in the previous subsection. In this subsection, we define a canonical family of almost hypercomplex structures on  $\hat{M}$  and consider their integrability.

Set  $e_0 := 1 \in \mathbb{R} (\cong T_1 \mathbb{R}^{>0})$  and  $Z_0^c := c \tilde{e}_0$  for a nonzero real number  $c$ . We denote the horizontal lifts relative to the connections  $\bar{\theta}$ ,  $\theta$ ,  $\theta_0$  by  $(\cdot)^{\bar{h}}$ ,  $(\cdot)^h$ ,  $(\cdot)^{h_0}$ , respectively. An almost hypercomplex structure  $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$  on  $\hat{M}$  is defined by

$$\hat{I}_\alpha^{\bar{\theta},c} Z_0^c = -Z_\alpha, \quad \hat{I}_\alpha^{\bar{\theta},c} Z_\alpha = Z_0^c, \quad \hat{I}_\alpha^{\bar{\theta},c} Z_\beta = Z_\gamma, \quad \hat{I}_\alpha^{\bar{\theta},c} Z_\gamma = -Z_\beta$$

and

$$(\hat{I}_\alpha^{\bar{\theta},c})_{(x,(\rho,s))}(X) = (I_\alpha(\hat{\pi}_* X))_{(x,(\rho,s))}^{\bar{h}}$$

for all horizontal vector  $X$  at  $(x, (\rho, s)) \in \hat{M}$ , where  $s = (I_1, I_2, I_3)$ . Note that the triple  $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$  depends on the connection form  $\bar{\theta}$  and  $c$ .

**Lemma 3.1.** *For any horizontal lift  $X^{\bar{h}} \in \bar{\mathcal{H}}_{(x,(\rho,s))}$  at  $(x, (\rho, s)) \in \hat{M}$ , we have  $(\Delta_{\#})_* X^{\bar{h}} = (X_\rho^{h_0}, X_s^h)$ . In particular, it holds  $(\Delta_{\#})_*((\hat{I}_\alpha^{\bar{\theta},c})_{(x,(\rho,s))}(X^{\bar{h}})) = ((I_\alpha X)_\rho^{h_0}, (I_\alpha X)_s^h)$ , where  $s = (I_1, I_2, I_3)$ . As a consequence, the horizontal lift  $X^{\bar{h}}$  of a vector field  $X$  on  $M$  is  $\Delta_{\#}$ -related to the vector field  $(X^{h_0}, X^h)$ , which is the horizontal lift of  $(X, X)$ :*

$$(\Delta_{\#})_* X^{\bar{h}} = (X^{h_0}, X^h) \circ \Delta_{\#}.$$

*Proof.*  $(\Delta_{\#})_* X^{\bar{h}}$  and  $(X^{h_0}, X^h)$  are horizontal vectors of  $S_0 \times S$ , since applying the connection form  $(\theta_0 \circ pr_{TS_0}, \theta \circ pr_{TS})$  on both vectors gives zero. On the other hand, applying  $(\pi_{S_0} \times \pi_S)_*$  on both vectors gives  $(X, X)$  because  $(\pi_{S_0} \times \pi_S) \circ \Delta_{\#} = \Delta \circ \hat{\pi}$ . This proves

$(\Delta_{\#})_* X^{\bar{h}} = (X^{h_0}, X^h) \circ \Delta_{\#}$ . Now it is easy to obtain  $(\Delta_{\#})_* ((\hat{I}_{\alpha}^{\bar{\theta},c})_{(x,(\rho,s))}(X^{\bar{h}})) = ((I_{\alpha}X)_{\rho}^{h_0}, (I_{\alpha}X)_s^h)$  using that  $(\hat{I}_{\alpha}^{\bar{\theta},c})_{(x,(\rho,s))}(X^{\bar{h}}) = (I_{\alpha}X)^{\bar{h}}$ .  $\square$

**Lemma 3.2.** *Let  $\nabla^1$  and  $\nabla^2 = \nabla^1 + S^{\xi}$  be quaternionic connections on  $(M, Q)$ , where  $\xi \in \Gamma(T^*M)$ . We denote the almost hypercomplex structure defined above with respect to  $\nabla^i$  ( $i = 1, 2$ ) and  $c \neq 0$  by  $\hat{I}_{\alpha}^{i,c}$  ( $\alpha = 1, 2, 3$ ). Then we have*

$$\hat{I}_{\alpha}^{1,c} - \hat{I}_{\alpha}^{2,c} = \varepsilon \left( 1 + \frac{4(n+1)}{c} \right) \left( (\hat{\pi}^* \xi) \otimes Z_{\alpha} + ((\hat{\pi}^* (\xi \circ I_{\alpha})) \otimes Z_0^c) \right)$$

at each point  $(x, (\rho, s)) \in \hat{M}$ , where  $s = (I_1, I_2, I_3)$ .

*Proof.* We consider any point  $(x, (\rho, s)) \in \hat{M}$ ,  $s = (I_1, I_2, I_3)$ , and omit the reference point in the proof. The corresponding connection forms induced by  $\nabla^i$  are denoted by  $\bar{\theta}^i$ ,  $\theta^i = (\theta_1^i, \theta_2^i, \theta_3^i)$ ,  $\theta_0^i$  ( $i = 1, 2$ ), respectively. The tangent bundle  $T\hat{M}$  is decomposed into  $T\hat{M} = \bar{\mathcal{V}} \oplus \bar{H}^1 = \bar{\mathcal{V}} \oplus \bar{H}^2$ , where  $\bar{H}^i = \text{Ker } \bar{\theta}^i$ . We express any tangent vector  $X$  of  $\hat{M}$  as

$$X = Y^{\bar{h}_i} + \sum_{\delta=1}^3 a_{\delta}^i Z_{\delta} + b^i Z_0^c,$$

where  $Y \in TM$ . By the definition of  $\hat{I}_{\alpha}^{i,c}$ , we see

$$\hat{I}_{\alpha}^{i,c}(X) = (I_{\alpha}Y)^{\bar{h}_i} + a_{\alpha}^i Z_0^c + a_{\beta}^i Z_{\gamma} - a_{\gamma}^i Z_{\beta} - b^i Z_{\alpha}.$$

Since

$$\begin{aligned} \bar{\theta}^1(X) &= \sum_{\delta=1}^3 a_{\delta}^1 e_{\delta} + cb^1 e_0 = \bar{\theta}^1(Y^{\bar{h}_2}) + \sum_{\delta=1}^3 a_{\delta}^2 e_{\delta} + cb^2 e_0 \\ &= \sum_{\delta=1}^3 \theta_{\delta}^1(\Delta_S Y^{\bar{h}_2}) e_{\delta} + \theta_0^1(\Delta_{S_0} Y^{\bar{h}_2}) e_0 + \sum_{\delta=1}^3 a_{\delta}^2 e_{\delta} + cb^2 e_0, \end{aligned}$$

we have  $b^1 = b^2 + (1/c)\theta_0^1(\Delta_{S_0} Y^{\bar{h}_2})$  and  $a_{\delta}^1 = a_{\delta}^2 + \theta_{\delta}^1(\Delta_S Y^{\bar{h}_2})$  ( $\delta = 1, 2, 3$ ). Therefore it holds

$$\begin{aligned} \hat{I}_{\alpha}^{1,c}(X) &= (I_{\alpha}Y)^{\bar{h}_1} + a_{\alpha}^1 Z_0^c + a_{\beta}^1 Z_{\gamma} - a_{\gamma}^1 Z_{\beta} - b^1 Z_{\alpha} \\ &= (I_{\alpha}Y)^{\bar{h}_1} + (a_{\alpha}^2 + \theta_{\alpha}^1(\Delta_S Y^{\bar{h}_2})) Z_0^c + (a_{\beta}^2 + \theta_{\beta}^1(\Delta_S Y^{\bar{h}_2})) Z_{\gamma} \\ &\quad - (a_{\gamma}^2 + \theta_{\gamma}^1(\Delta_S Y^{\bar{h}_2})) Z_{\beta} - (b^2 + (1/c)\theta_0^1(\Delta_{S_0} Y^{\bar{h}_2})) Z_{\alpha} \\ &= (I_{\alpha}Y)^{\bar{h}_1} - (I_{\alpha}Y)^{\bar{h}_2} + \hat{I}_{\alpha}^{2,c}(X) \\ &\quad + \theta_{\alpha}^1(\Delta_S Y^{\bar{h}_2}) Z_0^c + \theta_{\beta}^1(\Delta_S Y^{\bar{h}_2}) Z_{\gamma} - \theta_{\gamma}^1(\Delta_S Y^{\bar{h}_2}) Z_{\beta} - (1/c)\theta_0^1(\Delta_{S_0} Y^{\bar{h}_2}) Z_{\alpha}. \end{aligned}$$

Let  $s_0 : U \rightarrow S_0$  and  $s : U \rightarrow S$  be local sections defined on an open set  $U$  in  $M$ . Then  $\bar{s} := (s_0, s) \circ \Delta$  is a local section of  $\hat{M}$ . The pull backs of  $\theta^i$ ,  $\theta_0^i$  to  $U$  are denoted by  $\theta^{i,U}$  and  $\theta_0^{i,U}$ . If we define the one forms  $\theta_{\alpha}^{i,U}$  by  $\theta^{i,U} = s^* \theta^i = (1/2) \sum (\theta_{\alpha}^{i,U}) e_{\alpha}$ . From Lemma 2.1 and

$$\nabla^i I_{\alpha} = \varepsilon(\theta_{\gamma}^{i,U} \otimes I_{\beta} - \theta_{\beta}^{i,U} \otimes I_{\gamma}) \quad (i = 1, 2),$$

one can check that

$$(3.2) \quad \theta_{\delta}^{2,U} - \theta_{\delta}^{1,U} = -2\varepsilon(\xi \circ I_{\delta}) \quad (\delta = 1, 2, 3),$$

$$\theta_0^{2,U} - \theta_0^{1,U} = 4\varepsilon(n+1)\xi.$$

It is easy to see that

$$\begin{aligned}
 Y^{\bar{h}_1} - Y^{\bar{h}_2} &= \bar{s}_*(Y) - v_1(\bar{s}_*(Y)) - \bar{s}_*(Y) + v_2(\bar{s}_*(Y)) \\
 &= -\sum_{\delta=1}^3 (\theta_\delta^1(s_*Y) - \theta_\delta^2(s_*Y))Z_\delta - (1/c)(\theta_0^1(s_{0*}Y) - \theta_0^2(s_{0*}Y))Z_0^c \\
 &= -\frac{1}{2}\sum_{\delta=1}^3 (\theta_\delta^{1,U}(Y) - \theta_\delta^{2,U}(Y))Z_\delta - (1/c)(\theta_0^{1,U}(Y) - \theta_0^{2,U}(Y))Z_0^c \\
 (3.3) \quad &\stackrel{(3.2)}{=} -\varepsilon\sum_{\delta=1}^3 \xi(I_\delta Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(Y)Z_0^c,
 \end{aligned}$$

where  $v_i : T\hat{M} \rightarrow \bar{\mathcal{V}}$  is the projection with respect to  $\bar{\theta}^i$  ( $i = 1, 2$ ). Finally we obtain

$$\begin{aligned}
 \hat{I}_\alpha^{1,c}(X) - \hat{I}_\alpha^{2,c}(X) &= -\varepsilon\sum_{\delta=1}^3 \xi(I_\delta I_\alpha Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(I_\alpha Y)Z_0^c \\
 &\quad + \theta_\alpha^1(\Delta_S Y^{\bar{h}_2})Z_0^c + \theta_\beta^1(\Delta_S Y^{\bar{h}_2})Z_\gamma - \theta_\gamma^1(\Delta_S Y^{\bar{h}_2})Z_\beta - (1/c)\theta_0^1(\Delta_{S_0} Y^{\bar{h}_2})Z_\alpha \\
 &\stackrel{(*)}{=} -\varepsilon\sum_{\delta=1}^3 \xi(I_\delta I_\alpha Y)Z_\delta + \frac{4\varepsilon(n+1)}{c}\xi(I_\alpha Y)Z_0^c \\
 &\quad + \varepsilon\xi(I_\alpha Y)Z_0^c + \varepsilon\xi(I_\beta Y)Z_\gamma - \varepsilon\xi(I_\gamma Y)Z_\beta + \frac{4\varepsilon(n+1)}{c}\xi(Y)Z_\alpha \\
 &= \left(\varepsilon\xi(Y) + \frac{4(n+1)\varepsilon}{c}\xi(Y)\right)Z_\alpha + \left(\varepsilon\xi(I_\alpha Y) + \frac{4(n+1)\varepsilon}{c}\xi(I_\alpha Y)\right)Z_0^c,
 \end{aligned}$$

where in the step (\*) of the calculation we have computed

$$\theta_\alpha^1(\Delta_S Y^{\bar{h}_2}) = \theta_\alpha^1(Y^{h_2}) = \theta_\alpha^1(Y^{h_2} - Y^{h_1}) \stackrel{(3.3)}{=} \varepsilon\xi(I_\alpha Y)$$

and similarly for the other terms.  $\square$

The following proposition is an immediate consequence of Lemma 3.2, cf. the result with [21, Proposition 3.3].

**Proposition 3.3.** *The almost hypercomplex structure is independent of the choice of quaternionic connection if and only if  $c = -4(n+1)$ .*

Next we investigate transformation properties of the structures  $\hat{I}_\alpha^{\bar{\theta},c}$  ( $\alpha = 1, 2, 3$ ) under the principal action.

**Lemma 3.4.** *We have  $L_{Z_0}\hat{I}_\alpha^{\bar{\theta},c} = L_{Z_\alpha}\hat{I}_\alpha^{\bar{\theta},c} = 0$ ,  $L_{Z_\alpha}\hat{I}_\beta^{\bar{\theta},c} = 2\varepsilon\hat{I}_\gamma^{\bar{\theta},c}$  and  $L_{Z_\alpha}\hat{I}_\gamma^{\bar{\theta},c} = -2\varepsilon\hat{I}_\beta^{\bar{\theta},c}$ .*

*Proof.* Note first that the principal action generated by the vector fields  $Z_a$ ,  $a = 0, \dots, 3$ , preserves the horizontal and vertical distributions. Moreover, the central vector field  $Z_0$  commutes with the principal action and thus preserves the three canonical almost complex structures  $\hat{I}_\alpha^{\bar{\theta},c}$ .

Next we observe that it is easy to check the above equations on the vertical distribution

by evaluating them on  $Z_0^c, \dots, Z_3$ . So it only remains to check them on the horizontal distribution. Let  $\{\phi_t\}_{t \in \mathbb{R}}$  be the flow of  $Z_1$ . Since

$$\phi_t((x, (\rho, s))) = (x, (\rho, (I_1, (\cos 2\varepsilon t)I_2 + (\sin 2\varepsilon t)I_3, (-\sin 2\varepsilon t)I_2 + (\cos 2\varepsilon t)I_3)))$$

for  $(x, (\rho, s)) \in \hat{M}$ , where  $s = (I_1, I_2, I_3)$  and the horizontal lift of any vector field or tangent vector of  $M$  is invariant under  $\phi_t$ , we have

$$\begin{aligned} (L_{Z_1} \hat{I}_2^{\bar{\theta}, c})_{(x, (\rho, s))}(Y^h) &= [Z_1, \hat{I}_2^{\bar{\theta}, c} Y^h]_{(x, (\rho, s))} \\ &= \left. \frac{d}{dt} \phi_{t*}^{-1} ((\hat{I}_2^{\bar{\theta}, c} Y^h)_{\phi_t((x, (\rho, s)))}) \right|_{t=0} \\ &= \left. \frac{d}{dt} \phi_{t*}^{-1} ((\cos 2\varepsilon t)(I_2 Y)_{\phi_t((x, (\rho, s)))}^h + (\sin 2\varepsilon t)(I_3 Y)_{\phi_t((x, (\rho, s)))}^h) \right|_{t=0} \\ &= \left. \frac{d}{dt} ((\cos 2\varepsilon t)(I_2 Y)_{(x, (\rho, s))}^h + (\sin 2\varepsilon t)(I_3 Y)_{(x, (\rho, s))}^h) \right|_{t=0} \\ &= 2\varepsilon (I_3 Y)_{(x, (\rho, s))}^h = 2\varepsilon (\hat{I}_3^{\bar{\theta}, c})_{(x, (\rho, s))} Y^h \end{aligned}$$

and similarly  $L_{Z_1} \hat{I}_1^{\bar{\theta}, c} = 0$ , which imply  $L_{Z_1} \hat{I}_3^{\bar{\theta}, c} = -2\varepsilon \hat{I}_2^{\bar{\theta}, c}$ .  $\square$

The Nijenhuis tensor for  $\hat{I}_\alpha^{\bar{\theta}, c}$  is given by

$$N^\alpha(U, V) = [U, V] + \hat{I}_\alpha^{\bar{\theta}, c} [\hat{I}_\alpha^{\bar{\theta}, c} U, V] + \hat{I}_\alpha^{\bar{\theta}, c} [U, \hat{I}_\alpha^{\bar{\theta}, c} V] - [\hat{I}_\alpha^{\bar{\theta}, c} U, \hat{I}_\alpha^{\bar{\theta}, c} V]$$

for  $U, V \in \Gamma(T\hat{M})$ . Let  $\bar{\Omega}$  (resp.  $\Omega$ ) be the curvature form of  $\bar{\theta}$  (resp.  $\theta$ ). Take a local section  $s : U \rightarrow S$  defined on an open set  $U$  of  $M$ . The pull back of  $\Omega = \sum_{\alpha=1}^3 \Omega_\alpha e_\alpha$  by  $s$  is denoted by  $\Omega^U$ . Since the curvature form is horizontal, we have

$$(3.4) \quad \varepsilon \Omega|_{s(U)} = \pi_S^* \Omega^U|_{s(U)}.$$

If we define the two-forms  $\Omega_\alpha^U$  by  $\Omega^U = (1/2) \sum \Omega_\alpha^U e_\alpha$  and denote by  $\bar{\nabla}$  the connection on  $Q$  induced by  $\nabla$ , then we have

$$R_{X,Y}^{\bar{\nabla}} I_\alpha = [R_{X,Y}^{\bar{\nabla}}, I_\alpha] = \left[ \frac{1}{2} \sum \Omega_\delta^U(X, Y) I_\delta, I_\alpha \right] = \Omega_\gamma^U(X, Y) I_\beta - \Omega_\beta^U(X, Y) I_\gamma,$$

which implies

$$(3.5) \quad \Omega_\alpha^U(X, Y) = -\frac{1}{2n} \text{Tr} I_\alpha R_{X,Y}^{\bar{\nabla}}$$

for  $X, Y \in TM$ . In fact, multiplying the equation  $R_{X,Y}^{\bar{\nabla}} \circ I_\alpha - I_\alpha \circ R_{X,Y}^{\bar{\nabla}} = \Omega_\gamma^U(X, Y) I_\beta - \Omega_\beta^U(X, Y) I_\gamma$  with  $I_\beta$ , we obtain

$$I_\beta \circ R_{X,Y}^{\bar{\nabla}} \circ I_\alpha + I_\gamma \circ R_{X,Y}^{\bar{\nabla}} = -\Omega_\gamma^U(X, Y) \text{id} - \Omega_\beta^U(X, Y) I_\alpha.$$

Taking the trace proves (3.5). Let  $\text{Ric}^\nabla$  be the Ricci curvature of  $\nabla$  and its symmetric (resp. anti-symmetric) part is denoted by  $(\text{Ric}^\nabla)^s$  (resp.  $(\text{Ric}^\nabla)^a$ ). The Nijenhuis tensors of the canonical almost complex structures on the bundle  $\hat{M}$  over the quaternionic manifold  $(M, Q, \nabla)$  are computed in the next lemma.

**Lemma 3.5.** *If  $n > 1$  or  $Q$  is anti-self-dual provided  $n = 1$ , we have*

$$(3.6) \quad N^\alpha(Z_0^c, Z_i) = 0 \text{ for } 1 \leq i \leq 3,$$



$$(3.7) \quad N^\alpha(Z_i, Z_j) = 0 \text{ for } 1 \leq i, j \leq 3,$$

$$(3.8) \quad N^\alpha(Z_0^c, X^{\bar{h}}) = 0,$$

$$(3.9) \quad N^\alpha(Z_i, X^{\bar{h}}) = 0 \text{ for } 1 \leq i \leq 3,$$

$$(3.10) \quad \begin{aligned} & \bar{\theta}(N^\alpha(X^{\bar{h}}, Y^{\bar{h}})_{(x, (\rho, s))}) \\ &= \frac{4\varepsilon(n+1) + \varepsilon c}{2(n+1)} \left( (Ric^\nabla)^a(X, Y) - (Ric^\nabla)^a(I_\alpha X, I_\alpha Y) \right) e_0 \\ & \quad - \frac{4\varepsilon(n+1) + \varepsilon c}{2c(n+1)} \left( (Ric^\nabla)^a(X, I_\alpha Y) + (Ric^\nabla)^a(I_\alpha X, Y) \right) e_\alpha \text{ and} \end{aligned}$$

$$(3.11) \quad \hat{\pi}_*(N^\alpha(X^{\bar{h}}, Y^{\bar{h}})) = 0 \text{ for } X, Y \in \Gamma(TM),$$

where  $(x, (\rho, s)) \in \hat{M}$  ( $s = (I_1, I_2, I_3)$ ).

Proof. We write  $\hat{I}_\alpha = \hat{I}_\alpha^{\bar{\theta}, c}$  for simplicity in the proof of this lemma. It is easy to see that (3.6–3.9) hold by the definition of the almost hypercomplex structure on  $\hat{M}$  and Lemma 3.4. In fact, for example, we have  $N^\alpha(Z_\beta, Z_\gamma) = [Z_\beta, Z_\gamma] + [Z_\gamma, Z_\beta] = 0$  and  $N^\alpha(Z_\beta, X^{\bar{h}}) = \hat{I}_\alpha[Z_\beta, \hat{I}_\alpha X^{\bar{h}}] - [\hat{I}_\alpha Z_\beta, \hat{I}_\alpha X^{\bar{h}}] = \hat{I}_\alpha(-2\varepsilon \hat{I}_\gamma)(X^{\bar{h}}) - 2\varepsilon \hat{I}_\beta(X^{\bar{h}}) = 0$ . The other equations are proved similarly. Next we show (3.10). It holds  $(\theta_i \circ \hat{I}_\alpha)(Z_0^c) = -\delta_{i\alpha}$ ,  $(\theta_i \circ \hat{I}_\alpha)(Z_\alpha) = c\delta_{i0}$ ,  $(\theta_i \circ \hat{I}_\alpha)(Z_\beta) = \delta_{i\gamma}$ ,  $(\theta_i \circ \hat{I}_\alpha)(Z_\gamma) = -\delta_{i\beta}$ . Using this and Lemma 3.1, we have

$$\begin{aligned} & \bar{\theta}(\hat{I}_\alpha[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) \\ &= (\bar{\theta} \circ \hat{I}_\alpha) \left( \sum_{i=1}^3 \theta_i(\Delta_S[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) Z_i + \frac{1}{c} \theta_0(\Delta_{S_0}[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) Z_0^c \right) \\ &= \sum_{j=0}^3 \left( \sum_{i=1}^3 \theta_i(\Delta_S[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) (\theta_j \circ \hat{I}_\alpha)(Z_i) e_j + \frac{1}{c} \theta_0(\Delta_{S_0}[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) (\theta_j \circ \hat{I}_\alpha)(Z_0^c) e_j \right) \\ &= \sum_{j=0}^3 \left( \theta_\alpha(\Delta_S[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) c\delta_{j0} e_j + \theta_\beta(\Delta_S[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) \delta_{j\gamma} e_j \right. \\ & \quad \left. - \theta_\gamma(\Delta_S[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) \delta_{j\beta} e_j + \frac{1}{c} \theta_0(\Delta_{S_0}[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) (-\delta_{j\alpha}) e_j \right) \end{aligned}$$

As a consequence of Lemma 3.1, we have  $\Delta_{S_0}[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}] = [(I_\alpha X)^{h_0}, Y^{h_0}]|_{\Delta_\#(\hat{M})}$  and  $\Delta_S[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}] = [(I_\alpha X)^h, Y^h]|_{\Delta_\#(\hat{M})}$ . By  $\bar{\Omega} = \Omega + d\theta_0 = \sum_{\delta=1}^3 \Omega_\delta e_\delta + (d\theta_0)e_0$ , it holds

$$\begin{aligned} \bar{\theta}(\hat{I}_\alpha[\hat{I}_\alpha X^{\bar{h}}, Y^{\bar{h}}]) &= -c\Omega_\alpha((I_\alpha X)^h, Y^h)e_0 - \Omega_\beta((I_\alpha X)^h, Y^h)e_\gamma \\ & \quad + \Omega_\gamma((I_\alpha X)^h, Y^h)e_\beta + \frac{1}{c}d\theta_0((I_\alpha X)^{h_0}, Y^{h_0})e_\alpha. \end{aligned}$$

Defining

$$\begin{aligned} A_\alpha(X^{\bar{h}}, Y^{\bar{h}}) &= -\Omega_\beta(\Delta_S X^{\bar{h}}, \Delta_S Y^{\bar{h}}) + \Omega_\beta(\Delta_S \hat{I}_\alpha X^{\bar{h}}, \Delta_S \hat{I}_\alpha Y^{\bar{h}}) \\ & \quad + \Omega_\gamma(\Delta_S \hat{I}_\alpha X^{\bar{h}}, \Delta_S Y^{\bar{h}}) + \Omega_\gamma(\Delta_S X^{\bar{h}}, \Delta_S \hat{I}_\alpha Y^{\bar{h}}) \end{aligned}$$

for  $X, Y \in TM$ , we obtain

$$\begin{aligned}
& \bar{\theta}(N^\alpha(X^{\bar{h}}, Y^{\bar{h}})) \\
&= (-d\theta_0(X^{h_0}, Y^{h_0}) + d\theta_0((I_\alpha X)^{h_0}, (I_\alpha Y)^{h_0}) - c\Omega_\alpha(X^h, (I_\alpha Y)^h) - c\Omega_\alpha((I_\alpha X)^h, Y^h)e_0 \\
&\quad (-\Omega_\alpha(X^h, Y^h) + \Omega_\alpha((I_\alpha X)^h, (I_\alpha Y)^h) + \frac{1}{c}d\theta_0((I_\alpha X)^{h_0}, Y^{h_0}) + \frac{1}{c}d\theta_0(X^{h_0}, (I_\alpha Y)^{h_0}))e_\alpha \\
&\quad + A_\alpha(X^{\bar{h}}, Y^{\bar{h}})e_\beta + A_\alpha((I_\alpha X)^{\bar{h}}, Y^{\bar{h}})e_\gamma.
\end{aligned}$$

Next we show that the coefficients of  $e_0$  and  $e_\alpha$  can be described by the Ricci tensor of  $\nabla$  and that the other components vanish thanks to the integrability of the almost complex structure on the twistor space of  $M$  [22]. Set

$$(3.12) \quad B := \frac{1}{4(n+1)}(\text{Ric}^\nabla)^a + \frac{1}{4n}(\text{Ric}^\nabla)^s - \frac{1}{2n(n+2)}\Pi_h(\text{Ric}^\nabla)^s,$$

where  $\Pi_h(\text{Ric}^\nabla)^s$  is the  $Q$ -hermitian  $(0, 2)$ -tensor defined by

$$(\Pi_h(\text{Ric}^\nabla)^s)(X, Y) = \frac{1}{4} \left( (\text{Ric}^\nabla)^s(X, Y) + \sum_{i=1}^3 (\text{Ric}^\nabla)^s(I_i X, I_i Y) \right)$$

for  $X, Y \in TM$ . By [5], we have

$$(3.13) \quad \Omega_\alpha^U(X, Y) = 2(B(X, I_\alpha Y) - B(Y, I_\alpha X)).$$

Then it holds

$$\Omega_\alpha^U(I_\alpha X, Y) + \Omega_\alpha^U(X, I_\alpha Y) = -\frac{1}{n+1} \left( (\text{Ric}^\nabla)^a(X, Y) - (\text{Ric}^\nabla)^a(I_\alpha X, I_\alpha Y) \right).$$

Since  $\varepsilon d\theta_0^U(X, Y) = \text{Tr} R_{X,Y}^\nabla = -\text{Ric}^\nabla(X, Y) + \text{Ric}^\nabla(Y, X) = -2(\text{Ric}^\nabla)^a(X, Y)$  and  $\Omega_\alpha(X^h, Y^h) = (1/2)\varepsilon\Omega_\alpha^U(X, Y)$  for all tangent vector  $X, Y$  on  $M$ , to prove (3.10), it is sufficient to check  $A_\alpha = 0$ . This is related to the integrability of the almost complex structure on the twistor space  $\mathcal{Z}$  of the quaternionic manifold  $(M, Q)$  as we explain now. Recall that  $\mathcal{Z} = \{A \in Q \mid A^2 = -\text{id}\}$ . We set

$$R_{X,Y}^{\nabla(0,2)I} := \frac{1}{4}(R_{X,Y}^\nabla + IR_{IX,Y}^\nabla + IR_{X,IY}^\nabla - R_{IX,IY}^\nabla)$$

for  $X, Y \in TM$  and  $I \in \mathcal{Z}$ . Then

$$(3.14) \quad [R_{X,Y}^{\nabla(0,2)I}, I] = 0$$

for any  $I \in \mathcal{Z}$  if  $n > 1$ . In the case of  $\dim M = 4$ , (3.14) holds if and only if  $Q$  is anti-self-dual. See [3] for example. By (3.5) and (3.14), we have  $[R_{X,Y}^{\nabla(0,2)I_\alpha}, I_\alpha]I_\gamma = 0$  and thus

$$\begin{aligned}
0 &= 2\text{Tr}[R_{X,Y}^{\nabla(0,2)I_\alpha}, I_\alpha]I_\gamma \\
&= \text{Tr}(-I_\beta R_{X,Y}^\nabla + I_\beta R_{I_\alpha X, I_\alpha Y}^\nabla + I_\gamma R_{I_\alpha X, Y}^\nabla + I_\gamma R_{X, I_\alpha Y}^\nabla) \\
&= 2n(\Omega_\beta^U(X, Y) - \Omega_\beta^U(I_\alpha X, I_\alpha Y) - \Omega_\gamma^U(I_\alpha X, Y) - \Omega_\gamma^U(X, I_\alpha Y)) \\
&= -4n\varepsilon A_\alpha(X^{\bar{h}}, Y^{\bar{h}})
\end{aligned}$$

for all  $X, Y \in TM$ . This proves that  $A_\alpha = 0$ .

Since  $\nabla$  is torsion-free, we have (3.11) by the similar calculation for the Nijenhuis tensor of the almost complex structure on the twistor space.  $\square$

From Lemma 3.5 (and Proposition 3.3) we obtain the following result.

**Theorem 3.6.** *Let  $(M, Q)$  be a quaternionic manifold and  $\nabla$  a quaternionic connection. Let  $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$  be the almost hypercomplex structure on  $\hat{M}$ . We assume that  $Q$  is anti-self-dual when  $n = 1$ . If  $c = -4(n+1)$ , then the almost hypercomplex structure is integrable (and independent of  $\nabla$ ). When  $c \neq -4(n+1)$ , the almost hypercomplex structure is integrable if and only if  $(\text{Ric}^\nabla)^a$  is  $Q$ -hermitian, that is, it is hermitian with respect to  $I$  for all  $I \in \mathcal{Z}$ , where  $\mathcal{Z}$  is the twistor space of  $(M, Q)$ .*

We call  $\hat{M}$  the Swann bundle of  $M$ , although the terminology “Swann bundle” is also used for the quotient space  $\hat{M}/\mathbb{Z}$  with  $c = -4(n+1)$  in [20]. From now on we will only consider the case that  $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$  is a hypercomplex structure, i.e. integrable. We note that, for each fixed quaternionic connection,  $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c}) \neq (\hat{I}_1^{\bar{\theta},c'}, \hat{I}_2^{\bar{\theta},c'}, \hat{I}_3^{\bar{\theta},c'})$  if  $c \neq c'$ . Although it is obvious from the definition, we can also see it by considering the Obata connection. From Lemma 3.4, it follows that  $\hat{\nabla}_{\bar{e}_0}^c \bar{e}_0 = (1/c) \hat{\nabla}_{\bar{e}_0}^1 \bar{e}_0$ , where  $\hat{\nabla}^c$  is the Obata connection for the hypercomplex structure  $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ .

#### 4. A quaternionic vector field and its natural lift

A vector field  $X$  on  $(M, Q)$  is called *quaternionic* if its (local) flow  $\varphi_t$  satisfies

$$\varphi_{-t}^* I := \varphi_{t*} \circ I \circ \varphi_{t*}^{-1} \in Q$$

for all  $I \in Q$  and for all  $t$ . For a connection  $\nabla$  and  $X \in \Gamma(TM)$ , we define

$$(4.1) \quad (L_X \nabla)_Y Z := L_X(\nabla_Y Z) - \nabla_{L_X Y} Z - \nabla_Y(L_X Z),$$

where  $Y, Z \in \Gamma(TM)$ . Note that  $L_X \nabla$  is a tensor. In this paper, we study  $(M, Q)$  with a quaternionic vector field  $X$  which is also affine, that is  $L_X \nabla = 0$ . So we start by studying the condition  $L_X \nabla = 0$ . We define the Hessian  $H^\nabla$  with respect to  $\nabla$  by

$$H_{Y,Z}^\nabla X = \nabla_Y \nabla_Z X - \nabla_{\nabla_Y Z} X$$

for  $X, Y, Z \in \Gamma(TM)$ . By similar arguments as in [4], we have the following.

**Lemma 4.1.** *Let  $\nabla$  be a quaternionic connection of  $(M, Q)$  and  $X$  a quaternionic vector field. Then the following conditions are equivalent each other.*

- (1)  $L_X \nabla = 0$ ,
- (2)  $R_{X,Y}^\nabla Z = -H_{Y,Z}^\nabla X$  for all  $Y, Z \in TM$ ,
- (3)  $\text{Ric}^\nabla(X, Z) = \text{Tr} H_{(\cdot), Z}^\nabla X$  for all  $Z \in TM$ .

*Proof.* Since  $X$  is a quaternionic vector field,  $\varphi_{-t}^* \nabla$  is a quaternionic connection with respect to  $Q$ , where  $\varphi_{-t}^* \nabla$  is the connection defined by

$$(\varphi_{-t}^* \nabla)_Y Z = \varphi_{t*}(\nabla_{\varphi_{t*}^{-1} Y} \varphi_{t*}^{-1} Z)$$

for  $Y, Z \in \Gamma(TM)$ . Therefore there exists a one form  $\xi_t$  such that  $\varphi_t^* \nabla - \nabla = S^{\xi_t}$  by Lemma 2.1. Then we have

$$(4.2) \quad L_X \nabla = \left. \frac{d}{dt} \varphi_t^* \nabla \right|_{t=0} = \left. \frac{d}{dt} S^{\xi_t} \right|_{t=0} = S^{\xi_X},$$

where  $\xi_X = (d/dt)\xi_t|_{t=0}$ . On the other hand, by a straightforward calculation, we have

$(L_X \nabla)_Y Z = R_{X,Y}^\nabla Z + H_{Y,Z}^\nabla X$  for all  $Y, Z \in TM$ . Therefore, we have

$$(L_X \nabla)_Y Z = R_{X,Y}^\nabla Z + H_{Y,Z}^\nabla X = S_Z^{\xi_X} Y.$$

It follows that (1)  $\Rightarrow$  (2). It is also easy to see that (2)  $\Rightarrow$  (3) by taking a trace. Since  $\text{Tr} S_Z^{\xi_X} = 4(n+1)\xi_X(Z)$ , we have

$$-Ric^\nabla(X, Z) + \text{Tr} H_{(\cdot), Z}^\nabla X = \text{Tr} S_Z^{\xi_X} = 4(n+1)\xi_X(Z).$$

If (3) holds, then we have (1).  $\square$

We consider the normalizer

$$N(Q) := \{A \in \text{End}(TM) \mid [A, I] \in Q \text{ for all } I \in Q\}$$

and the centralizer

$$Z(Q) := \{A \in \text{End}(TM) \mid [A, I] = 0 \text{ for all } I \in Q\}.$$

Then we see  $N(Q) = Q + Z(Q) = Q + \mathbb{R} \cdot \text{id} + Z_0(Q)$ , where  $Z_0(Q)$  is the subspace of  $Z(Q)$  of trace-free tensors [5]. Let  $\nabla$  be a quaternionic connection and  $X$  a quaternionic vector field. Since  $L_X I_\alpha = \nabla_X I_\alpha + [I_\alpha, (\nabla X)]$ ,  $\nabla X$  is an element of  $N(Q)$ . We write  $\nabla X = T + T_0$ , where  $T \in \Gamma(Q + \mathbb{R} \cdot \text{id})$  and  $T_0 \in \Gamma(Z_0(Q))$ . Note that, by [5], we have explicitly

$$\begin{aligned} \nabla X &= -\frac{1}{4n} \sum_{\alpha=1}^3 \text{Tr}((\nabla X) \circ I_\alpha) I_\alpha \in \Gamma(Q) \\ &\quad + \frac{1}{4n} (\text{Tr} \nabla X) \text{id} \in C^\infty(M) \text{id} \\ &\quad + \frac{1}{4} ((\nabla X) - \sum_{\alpha=1}^3 I_\alpha (\nabla X) I_\alpha) - \frac{1}{4n} (\text{Tr} \nabla X) \text{id} \in \Gamma(Z_0(Q)). \end{aligned}$$

So it holds

$$(4.3) \quad T = \frac{1}{4n} \sum_{\alpha=0}^3 \varepsilon_\alpha \text{Tr}((\nabla X) \circ I_\alpha) I_\alpha,$$

where  $I_0 = \text{id}$  and  $\varepsilon_0 = 1, \varepsilon_1 = \varepsilon_2 = \varepsilon_3 = -1$ .

**Proposition 4.2.** *Let  $\nabla$  be a quaternionic connection and  $X$  a quaternionic vector field. Then  $L_X \nabla = 0$  if and only if  $2(Ric^\nabla)^a(X, \cdot) = d(\text{Tr}(\nabla X))$ .*

Proof. By the Bianchi identity, it holds

$$\begin{aligned} R_{X,Y}^\nabla Z &= -R_{Z,X}^\nabla Y - R_{Y,Z}^\nabla X \\ &= -R_{Z,X}^\nabla Y - H_{Y,Z}^\nabla X + H_{Z,Y}^\nabla X \\ &= -R_{Z,X}^\nabla Y - H_{Y,Z}^\nabla X + (\nabla_Z T)(Y) + (\nabla_Z T_0)(Y) \end{aligned}$$

for all  $Y, Z \in TM$ . Then we have  $-Ric^\nabla(X, Z) = -\text{Tr} R_{Z,X}^\nabla - \text{Tr} H_{(\cdot), Z}^\nabla X + \text{Tr}(\nabla_Z T)$ , since  $T_0$  and  $\nabla_Z T_0$  are trace-free. Therefore, by Lemma 4.1, we see that  $L_X \nabla = 0$  if and only if  $2(Ric^\nabla)^a(Z, X) + \text{Tr}(\nabla_Z T) = 0$ . Finally, because  $T \in \Gamma(Q + \mathbb{R} \cdot \text{id})$ , we obtain  $\text{Tr}(\nabla_Z T) = Z \text{Tr}(\nabla X)$  by (4.3). This implies the conclusion.  $\square$

Recall that every Killing vector field is affine with respect to the Levi-Civita connection. This means that every quaternionic Killing vector field  $X$  on a quaternionic Kähler manifold  $(M, g, Q)$  is an example of an affine quaternionic vector field. This can be seen also by Proposition 4.2.

**Corollary 4.3.** *Let  $X$  be a quaternionic vector field on a quaternionic manifold  $(M, Q)$ . If there exists a volume element  $\nu$  on  $M$  such that  $L_X \nu = 0$ , then there exists a quaternionic connection  $\nabla$  such that  $L_X \nabla = 0$ .*

*Proof.* We can find a quaternionic connection  $\nabla$  such that  $\nabla \nu = 0$  by [5, Theorem 2.4]. Then  $\text{Ric}^\nabla$  is symmetric. Because  $L_X = \nabla_X - (\nabla X)$  and  $L_X \nu = 0$ , we have  $\text{Tr}(\nabla X) = 0$ . Now the conclusion follows from Proposition 4.2.  $\square$

If  $X$  is a quaternionic vector field with the flow  $\{\varphi_t\}$ , then  $X$  can be lifted to  $\hat{X}$  on  $\hat{M}$  as follows. We define  $\hat{\varphi}_t : \hat{M} \rightarrow \hat{M}$  by

$$\hat{\varphi}_t((x, (\rho, s))) = (\varphi_t(x), (\varphi_{-t}^* \rho, (\varphi_{-t}^* I_1, \varphi_{-t}^* I_2, \varphi_{-t}^* I_3)))$$

for  $(x, (\rho, s)) \in \hat{M}$ , where  $s = (I_1, I_2, I_3)$  and define

$$\hat{X}_{(x, (\rho, s))} = \left. \frac{d}{dt} \hat{\varphi}_t((x, (\rho, s))) \right|_{t=0}.$$

The vector field  $\hat{X}$  on  $\hat{M}$  is called the *natural lift* of  $X$ . Since  $\hat{X}$  is invariant by the principal  $\mathbb{R}^{>0} \times \text{SO}(3)$ -action, we have the following.

**Lemma 4.4.** *Let  $\hat{X}$  be the natural lift of a quaternionic vector field  $X$ . We have  $[\hat{X}, \tilde{B}] = 0$  for  $B \in \mathbb{R} \oplus \mathfrak{so}(3)$ .*

Existence of  $\nu \in \Gamma(S_0)$  such that  $L_X \nu = 0$  (see Corollary 4.3) is related to the following condition for  $\hat{X}$ .

**Lemma 4.5.** *Let  $X$  be a quaternionic vector field  $X$  on  $(M, Q)$ . The following conditions are equivalent :*

- (1) *there exists  $\nu \in \Gamma(S_0)$  such that  $L_X \nu = 0$ ,*
- (2) *there exists a trivialization  $S_0 \cong M \times \mathbb{R}^{>0}$  such that  $\hat{X}_p \in T_p S \subset T_p \hat{M} \cong T_p S \oplus \mathbb{R}$  for all  $p = (x, (\rho, s)) \in \hat{M}$ .*

*Proof.* At first, assume that (1) holds. Then  $\nu$  gives a trivialization  $S_0 \cong M \times \mathbb{R}^{>0}$  and  $\hat{M} = S \times \mathbb{R}^{>0}$ . We denote the component of  $\hat{X}$  tangent to the second factor by  $\hat{X}^\mathbb{R}$ . For any point  $(x, (\rho, s)) \in \hat{M}$ , we see that  $\hat{X}_{(x, (\rho, s))}^\mathbb{R} = 0 \iff \hat{X}_{(x, (\nu(x), s))}^\mathbb{R} = 0 \iff (L_X \nu)_x = 0$  by Lemma 4.4. Conversely, we can obtain the desired section  $\nu \in \Gamma(S_0)$  by  $\nu(x) = \Phi^{-1}(x, 1)$  for each  $x \in M$ , where  $\Phi : S_0 \rightarrow M \times \mathbb{R}^{>0}$  is a trivialization satisfying (2).  $\square$

If there exists  $\nu \in \Gamma(S_0)$  such that  $L_X \nu = 0$ , we may assume that  $\hat{X}$  is a tangent vector field on  $S$  by Lemma 4.5. From now on we will assume that the quaternionic vector field  $X$  generates a free  $\text{U}(1)$ -action. Since  $\text{U}(1)$  is compact, there exists a volume form  $\nu$  invariant under the group action. This also implies that there exists a quaternionic connection  $\nabla$  such that  $L_X \nabla = 0$  by Corollary 4.3.

### 5. The hypercomplex moment map

In this section, we consider a hypercomplex moment map on the Swann bundle. In [14], a hypercomplex moment map is defined as follows.

**DEFINITION 5.1** [14]. Let  $M$  be a hypercomplex manifold with hypercomplex structure  $I_1, I_2, I_3$  and  $F$  a compact Lie group acting smoothly and freely on  $M$  preserving  $I_i$  ( $i = 1, 2, 3$ ).  $F$  acts on  $\mathfrak{F} = \text{Lie } F$  by the adjoint action. A vector field on  $M$  induced by  $f \in \mathfrak{F}$  is denoted by  $X_f$ . If a triple  $\mu = (\mu_1, \mu_2, \mu_3)$  of  $F$ -equivariant maps  $\mu_i : M \rightarrow \mathfrak{F}^*$  ( $i = 1, 2, 3$ ) satisfies

$$(5.1) \quad d\mu_1 \circ I_1 = d\mu_2 \circ I_2 = d\mu_3 \circ I_3$$

and

$$(5.2) \quad (d\mu_1 \circ I_1)(X_f) \text{ does not vanish on } M \text{ for any non-zero } f \in \mathfrak{F},$$

then  $\mu$  is called the *hypercomplex moment map* of  $F$ . The equations (5.1) are called the CR (Cauchy-Riemann) equations and the condition (5.2) is called the transversality condition.

A hypercomplex moment map produces another hypercomplex manifold by a quotient (Proposition 3.1 in [14]). Let  $(M, Q)$  be a quaternionic manifold with a quaternionic connection  $\nabla$  and an affine quaternionic vector field  $X$ . The following lemmas hold.

**Lemma 5.2.** *If  $X$  is an affine quaternionic vector field on  $(M, Q, \nabla)$  and  $\bar{\theta}$  is the principal  $\mathbb{R}^{>0} \times \text{SO}(3)$ -connection on  $\hat{M}$  induced by  $\nabla$ , then  $L_{\hat{X}}\bar{\theta} = 0$  and  $L_{\hat{X}}\hat{I}_\alpha^{\bar{\theta},c} = 0$ .*

*Proof.* The first equation follows from the fact that  $\hat{\varphi}_t$  preserves the horizontal distribution, because  $\hat{\varphi}_t$  is induced by a local flow  $\varphi_t$  of affine transformations preserving the quaternionic structure. Since the almost hypercomplex structure  $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$  is canonically associated with the data  $(Q, \nabla)$  on  $M$ , it is also invariant under  $\hat{\varphi}_t$ , which implies the second equation.  $\square$

From now on we assume that there exists  $\nu \in \Gamma(S_0)$  such that  $L_X\nu = 0$ . Then we can identify  $S_0 = M \times \mathbb{R}^{>0}$ ,  $\hat{M} = S \times \mathbb{R}^{>0}$  and  $\hat{X}$  is a tangent vector field on  $S$  by Lemma 4.5. In the next lemma, we identify  $S$  with the  $\hat{\varphi}_t$ -invariant submanifold  $S \times \{1\} \subset \hat{M} = S \times \mathbb{R}^{>0}$ .

**Lemma 5.3.** *Under the above assumption,  $L_{\hat{X}}\bar{\theta} = 0$ . Moreover  $\bar{\mathcal{H}}|_S = \mathcal{H}$  if and only if  $\nabla\nu = 0$ .*

*Proof.* The projection from  $\mathbb{R} \oplus \mathfrak{so}(3)$  onto  $\mathbb{R}$  (resp.  $\mathfrak{so}(3)$ ) is denoted by  $pr_{\mathbb{R}}$  (resp.  $pr_{\mathfrak{so}(3)}$ ). The first statement follows from the previous lemma, since  $pr_{\mathfrak{so}(3)}\bar{\theta}|_S = \theta$ . The second statement follows from  $pr_{\mathbb{R}}\bar{\theta}|_S = (\nu \circ \pi_S)^*\theta_0$ , since  $\nabla\nu = (\nu^*\theta_0) \otimes \nu$ .  $\square$

For  $1 \in \mathbb{R} \cong T_1\mathbb{R}^{>0}$ , at  $\rho \in S_0$ , we have

$$(\bar{e}_0)_\rho = \tilde{1}_\rho = \frac{d}{dr}\rho \exp(\varepsilon t) \Big|_{t=0} = \varepsilon\rho = \varepsilon r \frac{\partial}{\partial r} \Big|_\rho,$$

where  $r$  is the standard coordinate on  $\mathbb{R}^{>0}$ . Let  $\nabla$  be a quaternionic connection on  $(M, Q)$ . We define 1-forms  $\hat{\theta}_\alpha^c$  on  $\hat{M}$  ( $\alpha = 1, 2, 3$ ) by

$$\hat{\theta}_\alpha^c|_{TS} := Ar^{\frac{2}{c}}\theta_\alpha \quad \text{and} \quad \hat{\theta}_\alpha^c(Z_0^c) = 0$$

where  $A \in \mathbb{R}$  is a constant. A symmetric tensor  $\langle \theta, \theta \rangle$  is defined by

$$\langle \theta, \theta \rangle(Y, Z) = \sum_{i=1}^3 \theta_i(Y) \theta_i(Z)$$

for  $Y$  and  $Z \in TS$  and we set

$$(5.3) \quad G_\alpha^c := -Ar^{\frac{2}{c}} \Omega_\alpha(\cdot, \hat{I}_\alpha^{\bar{\theta}, c} \cdot) + 2\varepsilon Ar^{\frac{2}{c}} \langle \theta, \theta \rangle + \frac{2\varepsilon A}{c^2} r^{\frac{2}{c}-2} (dr \otimes dr).$$

Note that  $G_1^c|_{\mathcal{V} \times \mathcal{V}} = G_2^c|_{\mathcal{V} \times \mathcal{V}} = G_3^c|_{\mathcal{V} \times \mathcal{V}}$ , that is, the vertical components of  $G_\alpha^c$  are independent of  $\alpha$ .

**Lemma 5.4.** *We have  $d\hat{\theta}_\alpha^c(Y, Z) = G_\alpha^c(Y, \hat{I}_\alpha^{\bar{\theta}, c} Z)$  for  $Y, Z \in T\hat{M}$ .*

Proof. Put  $f(r) = Ar^{\frac{2}{c}}$ . Then

$$G_\alpha^c(Y, Z) = -f(r) \Omega_\alpha(Y, \hat{I}_\alpha^{\bar{\theta}, c} Z) + 2\varepsilon f(r) \langle \theta, \theta \rangle(Y, Z) + \frac{2\varepsilon f(r)}{c^2} \frac{f(r)}{r^2} (dr \otimes dr)(Y, Z)$$

for  $Y, Z \in T\hat{M}$ . Since  $Z_0^c = \varepsilon cr \frac{\partial}{\partial r}$ , we obtain

$$d\hat{\theta}_\alpha^c(Z_0^c, Z_\alpha) = \varepsilon cr f'(r) = \varepsilon cr \cdot \frac{2A}{c} r^{\frac{2}{c}-1} = 2\varepsilon f(r),$$

$$G_\alpha^c(Z_0^c, \hat{I}_\alpha^{\bar{\theta}, c}(Z_\alpha)) = G_\alpha^c(Z_0^c, Z_0^c) = c^2 r^2 \cdot \frac{2\varepsilon f(r)}{c^2} \frac{f(r)}{r^2} = 2\varepsilon f(r)$$

and

$$G_\alpha^c(Z_\alpha, \hat{I}_\alpha^{\bar{\theta}, c}(Z_0^c)) = -G_\alpha^c(Z_\alpha, Z_\alpha) = -2\varepsilon f(r).$$

Moreover we have

$$d\hat{\theta}_\alpha^c(Z_\beta, Z_\gamma) = -\hat{\theta}_\alpha^c([Z_\beta, Z_\gamma]) = -2\varepsilon f(r)$$

and

$$G_\alpha^c(Z_\beta, \hat{I}_\alpha^{\bar{\theta}, c}(Z_\gamma)) = -G_\alpha^c(Z_\beta, Z_\beta) = -2\varepsilon f(r),$$

similarly  $G_\alpha^c(Z_\gamma, \hat{I}_\alpha^{\bar{\theta}, c}(Z_\beta)) = 2\varepsilon f(r)$ . Finally, we see

$$d\hat{\theta}_\alpha^c(Y^h, Z^h) = f(r)(d\theta_\alpha^c)(Y^h, Z^h) = f(r)\Omega_\alpha(Y^h, Z^h)$$

and

$$G_\alpha^c(Y^h, \hat{I}_\alpha^{\bar{\theta}, c} Z^h) = f(r)\Omega_\alpha(Y^h, Z^h).$$

For other combinations of tangent vectors on  $\hat{M}$ , both tensors  $d\hat{\theta}_\alpha^c, G_\alpha^c$  vanish.  $\square$

We define  $\mu^c : \hat{M} \rightarrow \mathbb{R}^3$  by

$$(5.4) \quad \mu^c(x) = \hat{\theta}^c(\hat{X}_x) = (\hat{\theta}_1^c(\hat{X}_x), \hat{\theta}_2^c(\hat{X}_x), \hat{\theta}_3^c(\hat{X}_x))$$

for  $x \in \hat{M}$ . We calculate some formulae which will be used later to determine sufficient conditions for  $\mu^c$  to be a hypercomplex moment map.

**Lemma 5.5.** *If  $X$  is an affine quaternionic vector field on  $(M, Q, \nabla)$ , then we have*

$$d\mu_\alpha^c = -\iota_{\hat{X}} d\hat{\theta}_\alpha^c.$$

Proof. By Lemmas 4.4 and 5.3,  $L_{\hat{X}}\theta_\alpha = 0$  and  $[\hat{X}, Z_0^c] = 0$ . It follows  $L_{\hat{X}}\hat{\theta}_\alpha^c = 0$  and  $d\mu_\alpha^c = d\iota_{\hat{X}}\hat{\theta}_\alpha^c = L_{\hat{X}}\hat{\theta}_\alpha^c - \iota_{\hat{X}}d\hat{\theta}_\alpha^c = -\iota_{\hat{X}}d\hat{\theta}_\alpha^c$ .  $\square$

For the CR-condition for  $\mu^c$ , we have

**Lemma 5.6.** *If  $X$  is an affine quaternionic vector field on  $(M, Q, \nabla)$ , then we have*

$$(5.5) \quad (d\mu_\alpha^c \circ \hat{I}_\alpha^{\bar{\theta},c})(Z_0^c) = 0,$$

$$(5.6) \quad (d\mu_1^c \circ \hat{I}_1^{\bar{\theta},c})(\tilde{B}) = (d\mu_2^c \circ \hat{I}_2^{\bar{\theta},c})(\tilde{B}) = (d\mu_3^c \circ \hat{I}_3^{\bar{\theta},c})(\tilde{B}) \text{ for any } B \in \mathfrak{so}(3),$$

$$(5.7) \quad (d\mu_\alpha^c \circ \hat{I}_\alpha^{\bar{\theta},c})(Y) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}_\alpha^{\bar{\theta},c}Y),$$

for all horizontal vector  $Y$ .

Proof. By Lemmas 5.4 and 5.5, we have  $d\mu_\alpha^c \circ \hat{I}_\alpha^{\bar{\theta},c} = G_\alpha^c(\hat{X}, \cdot)$ . Then it is easy to see  $(d\mu_\alpha^c \circ \hat{I}_\alpha^{\bar{\theta},c})(Z_0^c) = G_\alpha^c(\hat{X}, Z_0^c) = 0$ . Since  $G_1^c = G_2^c = G_3^c$  on  $\tilde{\mathcal{V}} \times T\hat{M}$ , we obtain (5.6). Finally for horizontal vector  $Y$  we have  $(d\mu_\alpha^c \circ \hat{I}_\alpha^{\bar{\theta},c})(Y) = G_\alpha^c(\hat{X}, Y) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}_\alpha^{\bar{\theta},c}Y)$ .  $\square$

For the transversality condition for  $\mu^c$ , we state the next lemma, which follows from the equation  $d\mu_\alpha^c \circ \hat{I}_\alpha^{\bar{\theta},c} = G_\alpha^c(\hat{X}, \cdot)$ .

**Lemma 5.7.** *We have*

$$(d\mu_\alpha^c \circ \hat{I}_\alpha^{\bar{\theta},c})(\hat{X}) = G_\alpha^c(\hat{X}, \hat{X}) = -Ar^{\frac{2}{c}}\Omega_\alpha(\hat{X}, \hat{I}_\alpha^{\bar{\theta},c}\hat{X}) + 2\varepsilon Ar^{\frac{2}{c}}\langle \theta, \theta \rangle(\hat{X}, \hat{X}).$$

By Lemma 5.6,  $\mu^c$  satisfies CR equations (5.1) if and only if

$$\Omega_1(\hat{X}, \hat{I}_1^{\bar{\theta},c}Y) = \Omega_2(\hat{X}, \hat{I}_2^{\bar{\theta},c}Y) = \Omega_3(\hat{X}, \hat{I}_3^{\bar{\theta},c}Y)$$

holds for all horizontal vector  $Y$ . On the other hand, from the equations (3.12) and (3.13),  $\Omega_\alpha$  satisfies

$$(5.8) \quad \begin{aligned} 2\varepsilon\Omega_\alpha(Y^h, \hat{I}_\alpha^{\bar{\theta},c}Z^h) &= \Omega_\alpha^U(Y, I_\alpha Z) \\ &= \frac{1}{2(n+1)} \left( (Ric^\nabla)^a(I_\alpha Y, I_\alpha Z) - (Ric^\nabla)^a(Y, Z) \right) \\ &\quad - \frac{1}{2n} \left( (Ric^\nabla)^s(I_\alpha Y, I_\alpha Z) + (Ric^\nabla)^s(Y, Z) \right) \\ &\quad + \frac{2}{n(n+2)} (\Pi_h(Ric^\nabla)^s)(Y, Z) \end{aligned}$$

for tangent vectors  $Y$  and  $Z$  on  $M$ . In particular, if  $Ric^\nabla$  is  $Q$ -hermitian, then  $\Omega_\alpha(\cdot, \hat{I}_\alpha^{\bar{\theta},c}\cdot)$  does not depend on  $\alpha$ . We see that  $\mu^c$  satisfies the CR equations (5.1) if  $Ric^\nabla(X, Y) = Ric^\nabla(IX, IY)$  for all  $Y \in TM$  and  $I \in \mathcal{Z}$ . Moreover if the vector field  $X$  on  $M$  is affine quaternionic and  $\mu^c$  satisfies the CR equations, then

$$\Omega_\alpha(\hat{X}, \hat{I}_\alpha^{\bar{\theta},c}\hat{X}) = \Omega_\beta(\hat{X}, \hat{I}_\beta^{\bar{\theta},c}\hat{X}) = \Omega_\gamma(\hat{X}, \hat{I}_\gamma^{\bar{\theta},c}\hat{X}) = -\frac{\varepsilon}{2(n+2)}(Ric^\nabla)(X, X) \circ \hat{\pi}.$$

The following statements can be obtained for the CR equations and the transversality con-



dition.

**Proposition 5.8.** *Let  $M$  be a quaternionic manifold with a quaternionic connection  $\nabla$  and  $X$  an affine quaternionic vector field on  $(M, Q, \nabla)$ . Assume that there exists  $v \in \Gamma(S_0)$  such that  $L_X v = 0$ . If*

$$\text{Ric}^\nabla(X, Y) = \text{Ric}^\nabla(IX, IY)$$

for all  $Y \in TM$ ,  $I \in \mathcal{Z}$  and

$$(\text{Ric}^\nabla)(X, X) \circ \hat{\pi} + 4(n+2)\langle \theta, \theta \rangle(\hat{X}, \hat{X})$$

does not vanish on  $\hat{M}$ , then the map  $\mu^c = \text{Ar}^{\frac{2}{c}}\theta(\hat{X}) : \hat{M} \rightarrow \mathbb{R}^3$  ( $A \neq 0$ ) satisfies the CR equations and the transversality condition for any  $c \neq 0$ .

Also we have

**Corollary 5.9.** *Let  $M$  be a quaternionic manifold with a quaternionic connection  $\nabla$  and  $X$  an affine quaternionic vector field on  $(M, Q, \nabla)$ . Assume that there exists  $v \in \Gamma(S_0)$  such that  $L_X v = 0$ . If  $\text{Ric}^\nabla$  is  $Q$ -hermitian and*

$$(\text{Ric}^\nabla)(X, X) \circ \hat{\pi} + 4(n+2)\langle \theta, \theta \rangle(\hat{X}, \hat{X})$$

does not vanish on  $\hat{M}$ , then the map  $\mu^c = \text{Ar}^{\frac{2}{c}}\theta(\hat{X}) : \hat{M} \rightarrow \mathbb{R}^3$  ( $A \neq 0$ ) satisfies the CR equations and the transversality condition for any  $c \neq 0$ .

## 6. The proof of the main result

In this section, we give the proof of our main result. Using the hypercomplex quotient in [14], we can obtain a hypercomplex manifold  $M'$  with certain properties. To show it, the following lemmas are needed.

**Lemma 6.1.** *We have  $L_{\tilde{B}}\theta = -\varepsilon[B, \theta]$ . Moreover  $L_{Z_\alpha}\theta_\alpha = 0$ ,  $L_{Z_\alpha}\theta_\beta = 2\varepsilon\theta_\gamma$  and  $L_{Z_\alpha}\theta_\gamma = -2\varepsilon\theta_\beta$ .*

*Proof.* We see that  $(L_{\tilde{B}}\theta)(\tilde{C}) = -\theta([\tilde{B}, \tilde{C}]) = -\varepsilon[B, C] = -\varepsilon[B, \theta(\tilde{C})]$  and  $(L_{\tilde{B}}\theta)(Y^h) = -\theta([\tilde{B}, Y^h]) = 0 (= -[B, \theta(Y^h)])$ . For the latter statements, we compute

$$\sum (L_{Z_\alpha}\theta_i)e_i = L_{Z_\alpha}\theta = -\varepsilon[e_\alpha, \sum \theta_i e_i] = -\varepsilon[e_\alpha, \theta_\beta e_\beta] - \varepsilon[e_\alpha, \theta_\gamma e_\gamma] = -2\varepsilon\theta_\beta e_\gamma + 2\varepsilon\theta_\gamma e_\beta. \quad \square$$

By Lemma 6.1, we have

**Lemma 6.2.** *It holds that  $L_{Z_\alpha}\hat{\theta}_\alpha^c = 0$ ,  $L_{Z_\alpha}\hat{\theta}_\beta^c = 2\varepsilon\hat{\theta}_\gamma^c$  and  $L_{Z_\alpha}\hat{\theta}_\gamma^c = -2\varepsilon\hat{\theta}_\beta^c$ .*

From Lemma 5.3, it holds

**Lemma 6.3.** *If  $L_X \nabla = 0$ , then  $L_{\hat{X}}d\hat{\theta}_\alpha^c = 0$  for  $\alpha = 1, 2, 3$ .*

We can now prove the main theorems in this paper.

**Theorem 6.4.** *Let  $(M, Q)$  be a quaternionic manifold. We assume that  $Q$  is anti-self-dual when  $n = 1$ . Moreover assume that  $U(1)$  acts freely on  $M$  preserving  $Q$ . We denote by  $X$  the vector field generating the  $U(1)$ -action. If  $Q$  admits a quaternionic connection  $\nabla$  such that*

$$L_X \nabla = 0,$$

$$(6.1) \quad Ric^\nabla(X, Y) = Ric^\nabla(IX, IY)$$

for all  $Y \in TM$ ,  $I \in \mathcal{Z}$  and

$$(6.2) \quad (Ric^\nabla)(X, X) \circ \hat{\pi} + 4(n+2)\langle \theta, \theta \rangle (\hat{X}, \hat{X})$$

does not vanish on  $\hat{M}$ , then the natural lift  $\hat{X}$  generates a free  $U(1)$ -action with the moment map  $\mu^c$  defined by (5.4), where  $c = -4(n+1)$ . Then the corresponding hypercomplex quotient is a hypercomplex manifold  $(M', H = (I'_1, I'_2, I'_3))$  with an  $I'_1$ -holomorphic vector field  $Z$  such that  $L_Z I'_2 = 2\varepsilon I'_3$ ,  $L_Z I'_3 = -2\varepsilon I'_2$ . Moreover the exact 2-forms  $d\hat{\theta}_\alpha^c$  on  $\hat{M}$  induce closed 2-forms  $\Theta'_\alpha$  on  $M'$  which satisfy  $L_Z \Theta'_1 = 0$ ,  $L_Z \Theta'_2 = 2\varepsilon \Theta'_3$ ,  $L_Z \Theta'_3 = -2\varepsilon \Theta'_2$ .

Proof. Choose  $c = -4(n+1)$ . Then  $\hat{M}$  is a hypercomplex manifold by Theorem 3.6. We can choose a  $U(1)$ -invariant volume form  $\nu$  on  $M$ . Then the condition (2) in Lemma 4.5 holds, so  $\hat{X}$  is tangent to  $S$ , which means that the results in the previous section can be applied. Since Proposition 5.8 and the second statement of Lemma 5.2 hold,  $M' = P/U(1)$  is a hypercomplex manifold with the induced hypercomplex structure  $I'_1, I'_2, I'_3$  by [14, Proposition 3.1], where  $P$  is the level set  $(\mu^c)^{-1}((1, 0, 0))$ . Based on the proof of [14, Proposition 3.1], take  $V = \{v \in TP \mid (d\mu_\alpha^c \circ \hat{I}_\alpha^{c,c})(v) = 0\}$ . Then we see that  $TP = V \oplus \langle \hat{X} \rangle$ . In particular,  $\pi_P^* TM' \cong V$ , where  $\pi_P : P \rightarrow M'$  is the quotient map. The vector field  $\hat{I}_1^{c,c} Z_0^c = Z_1$  is tangent to  $P$ , since

$$Z_1 \mu_\alpha^c|_P = (2\varepsilon \delta_{2\alpha} \mu_3^c - 2\varepsilon \delta_{3\alpha} \mu_2^c)|_P$$

by Lemma 6.2. By Lemma 4.4,  $Z_1$  is a projectable vector field, that is,  $Z := \pi_{P*}(Z_1)$  is a vector field on  $M'$ . The vector field  $Z$  satisfies  $(L_Z I'_\alpha)(U) = \pi_{P*}((L_{Z_1} \hat{I}_\alpha^{c,c})(U_P))$ , where  $U_P \in \Gamma(V)$  is any projectable vector field and  $U = \pi_{P*}(U_P)$  is its projection. In fact, this can be obtained from  $I'_\alpha \circ \pi_{P*} \circ pr_V = \pi_{P*} \circ pr_V \circ \hat{I}_\alpha^{c,c}$ , where  $pr_V : T\hat{M}|_P \rightarrow V$  is the projection with respect to the  $\hat{I}_\alpha^{c,c}$ -invariant decomposition

$$(T\hat{M})|_P = V \oplus \langle \hat{X}, \hat{I}_1^{c,c} \hat{X}, \hat{I}_2^{c,c} \hat{X}, \hat{I}_3^{c,c} \hat{X} \rangle.$$

Therefore, by Lemmas 3.4 and 4.4, we see that  $Z$  is a  $I'_1$ -holomorphic vector field such that  $L_Z I'_2 = 2\varepsilon I'_3$  and  $L_Z I'_3 = -2\varepsilon I'_2$ . Finally, by Lemma 6.3, we can define 2-forms  $\Theta'_1, \Theta'_2, \Theta'_3$  on  $M'$  by  $\Theta'_\alpha(U, W) = (d\hat{\theta}_\alpha^c)(U_P, W_P)$  for  $U = \pi_{P*}(U_P)$  and  $W = \pi_{P*}(W_P)$ . It is clear that these forms are closed. Finally we see that these forms satisfy the desired conditions by Lemma 6.2.  $\square$

**REMARK 6.5.** In Theorem 6.4, the same conclusion can be obtained under the assumption that the action induced by  $\hat{X}$  is free instead of the assumption that the action induced by  $X$  is free.

In the case that  $Ric^\nabla$  is  $Q$ -hermitian, we have the following theorem.

**Theorem 6.6.** *Let  $(M, Q)$  be a quaternionic manifold. We assume that  $Q$  is anti-self-dual when  $n = 1$ . Moreover assume that  $U(1)$  acts freely on  $M$  preserving  $Q$ . We denote by  $X$  the vector field generating the  $U(1)$ -action. If  $Q$  admits a quaternionic connection  $\nabla$  such that*

$L_X \nabla = 0$ ,  $\text{Ric}^\nabla$  is  $Q$ -hermitian and

$$(6.3) \quad (\text{Ric}^\nabla)(X, X) \circ \hat{\pi} + 4(n+2)\langle \theta, \theta \rangle (\hat{X}, \hat{X})$$

does not vanish on  $\hat{M}$ , then there exists a 1-parameter family  $\{(M'^c, H^c = (I_1'^c, I_2'^c, I_3'^c))\}_{c \neq 0}$  of hypercomplex manifolds with an  $I_1'^c$ -holomorphic vector field  $Z^c$  on  $M'^c$  such that  $L_{Z^c} I_2'^c = 2\varepsilon I_3'^c$ ,  $L_{Z^c} I_3'^c = -2\varepsilon I_2'^c$ . Moreover the exact 2-forms  $d\hat{\theta}_\alpha^c$  on  $\hat{M}$  give a 1-parameter family  $\{(\Theta_1'^c, \Theta_2'^c, \Theta_3'^c)\}_{c \neq 0}$  of triplets of closed 2-forms on  $M'^c$  such that  $L_{Z^c} \Theta_1'^c = 0$ ,  $L_{Z^c} \Theta_2'^c = 2\varepsilon \Theta_3'^c$ ,  $L_{Z^c} \Theta_3'^c = -2\varepsilon \Theta_2'^c$  and

$$(6.4) \quad \Theta_1'^c(\cdot, I_1'^c \cdot) = \Theta_2'^c(\cdot, I_2'^c \cdot) = \Theta_3'^c(\cdot, I_3'^c \cdot).$$

Proof. Since  $\text{Ric}^\nabla$  is  $Q$ -hermitian, the almost hypercomplex structures  $(\hat{I}_1^{\hat{\theta},c}, \hat{I}_2^{\hat{\theta},c}, \hat{I}_3^{\hat{\theta},c})$  on  $\hat{M}$  are integrable for all  $c \neq 0$  by Theorem 3.6. Following the same procedure as in the proof of Theorem 6.4, we obtain the claims with exception of the equation (6.4). The latter equation follows from  $Q$ -hermitian assumption for  $\text{Ric}^\nabla$  using Lemma 5.4 and (5.8).  $\square$

The assumption (6.2) is formulated in terms of objects on the Swann bundle  $\hat{M}$ . We have the following corollary under assumptions formulated directly on  $M$ .

**Corollary 6.7.** *Let  $(M, Q)$  be a quaternionic manifold. We assume that  $Q$  is anti-self-dual when  $n = 1$ . Moreover assume that  $U(1)$  acts freely on  $M$  preserving  $Q$ . We denote by  $X$  the vector field generating the  $U(1)$ -action. If  $Q$  admits a quaternionic connection  $\nabla$  such that  $L_X \nabla = 0$ ,  $\text{Ric}^\nabla(X, X) > 0$  and (6.1) is satisfied (resp.  $\text{Ric}^\nabla$  is  $Q$ -hermitian), then we have the same conclusion as Theorem 6.4 (resp. Theorem 6.6).*

We call the correspondence from  $(M, Q, X)$  to  $(M', H, Z)$  or to  $\{(M'^c, H^c, Z^c)\}_{c \neq 0}$  described in Theorems 6.4 and 6.6 the *Quaternionic/Hypercomplex-correspondence* ( $Q/H$ -correspondence for short).

**A relation with Swann's twist construction.** Now we explain how  $M'$  considered just as a smooth manifold can be related to  $M$  by Swann's twist construction. Consider the Lie subgroup  $U(1)_{Z_1} := \{g \in \text{SO}(3) \mid \text{Ad}_g e_1 = e_1\}$  of  $\text{SO}(3)$ , which can be identified with  $U(1)$ . Notice this group is different from the group  $\langle \hat{X} \rangle \cong U(1)$  generated by  $\hat{X}$ . Then  $P = (\mu^c)^{-1}((1, 0, 0)) = (\mu^c)^{-1}(e_1)$  is a principal  $U(1)_{Z_1}$ -bundle over  $\hat{\pi}(P)$  with a connection  $\iota_P^*(\theta_1)$ , where  $\iota_P : P \rightarrow \hat{M}$  is the inclusion map from  $P$ . In fact, the calculation

$$\mu^c(pg) = \hat{\theta}(\hat{X}_{pg}) = \hat{\theta}(R_{g*} \hat{X}_p) = \text{Ad}_{g^{-1}} \hat{\theta}(\hat{X}_p) = \text{Ad}_{g^{-1}} e_1 = e_1$$

for  $p \in P$  and  $g \in U(1)_{Z_1}$  shows that  $P$  is invariant under  $U(1)_{Z_1}$ . In particular,  $P \cap \hat{\pi}^{-1}(x)$  is a union of circles ( $U(1)_{Z_1}$ -orbits). Since the functions  $\theta_\alpha(\hat{X})|_{\hat{\pi}^{-1}(x)}$  on  $\hat{\pi}^{-1}(x) \cong \mathbb{H}^*/\{\pm 1\}$  are linear in the natural coordinates on  $\mathbb{H} \cong \mathbb{R}^4$  and  $\hat{\theta}_\alpha(\hat{X}) = \text{Ar}^{\frac{2}{c}} \theta_\alpha(\hat{X})$ , we see that the above intersection  $P \cap \hat{\pi}^{-1}(x)$  is a single circle. Recall [24] that Swann's twist construction produces a new manifold  $M'$  from a manifold  $M$  with the following twist data: a vector field  $\xi$ , a two form  $F$  and a function  $a$  on  $M$ . More precisely,  $\xi$  generates a  $U(1)$ -action,  $F$  is an invariant closed 2-form which is the curvature form of a connection form on a principal  $U(1)$ -bundle, and  $a$  is non-vanishing and satisfies  $da = -\iota_\xi F$ . It was shown in [17] that the HK/QK-correspondence can be described using the twist construction and a so-called elementary deformation of the metric.

In the setting of the  $Q/H$ -correspondence, let  $s : U \rightarrow P$ ,  $U \subset \hat{\pi}(P)$  be a local section.

Then we define a two form  $F$  and a function  $a$  on  $\hat{\pi}(P)$  by

$$\begin{aligned} F &:= s^*(d(\iota_P^* \theta_1)) = s^*(d\theta_1) = s^* \Omega_1, \\ a &:= s^*(\theta_1(\hat{X}) \circ \iota_P) = \theta_1(\hat{X}) \circ s_* = s^*(\theta_1(\hat{X})). \end{aligned}$$

Note that both  $F$  and  $a$  are independent of the choice of  $s$ . Then we have

**Proposition 6.8.** *As a smooth manifold,  $M'$  obtained by the  $Q/H$ -correspondence is a twist of  $\hat{\pi}(P)$  in the sense of [24] with the twist data  $(\xi = X, F, a)$  as above.*

Proof. Since  $L_X \nabla = 0$ , we have

$$\begin{aligned} da &= s^*(d\iota_{\hat{X}} \theta_1) = -s^*(d\theta_1(\hat{X}, \cdot)) = -\Omega_1(\hat{X}, s_*(\cdot)) \\ &= -\Omega_1(X^h, s_*(\cdot)) = -\Omega_1(s_*(X), s_*(\cdot)) = -F(X, \cdot). \end{aligned}$$

Also we obtain  $L_X F = (\iota_X d + d\iota_X)F = -dda = 0$ .  $\square$

Note that the complex structures  $I'_\alpha$  are not  $\mathcal{H}$ -related to  $I_\alpha$  in the sense of [24], because the invariant subbundle  $V \subset T\hat{M}$  does not coincides with  $\mathcal{H}$  in general.

## 7. Examples

In this section, we give examples.

**$QK/HK$ -correspondence:** When  $M$  is a possibly indefinite quaternionic Kähler manifold with non zero scalar curvature, we can take the Levi-Civita connection  $\nabla$  as a quaternionic connection and if there exists a non-zero quaternionic Killing vector field  $X$  on  $M$ , then we can take  $X$  as the affine quaternionic vector field in the  $Q/H$ -correspondence. The tensor field (5.3) gives a (pseudo-)hyper-Kähler metric on  $\hat{M}$  and (6.4) gives a (pseudo-)hyper-Kähler metric on  $M'$  if  $\hat{X}$  is time-like or space-like (see [1]). Therefore our  $Q/H$ -correspondence is a generalization of the  $QK/HK$ -correspondence. The following example is well-known (see [13, 11, 23] for example).

**EXAMPLE 7.1 THE COTANGENT BUNDLE  $T^*\mathbb{C}P^n$  AS A HYPER-KÄHLER MANIFOLD.** Consider the quaternionic (right-)projective space  $M = \mathbb{H}P^n$  with the standard quaternionic structure. We can choose the Levi-Civita connection  $\nabla$  of the standard quaternionic Kähler metric on  $M$  as a quaternionic connection. Then we see the Swann bundle  $\hat{M} = (\mathbb{H}^{n+1} \setminus \{0\})/\{\pm 1\} \rightarrow M$  as a hypercomplex manifold with the hypercomplex structure  $(\hat{I}_1^{\bar{\theta},c}, \hat{I}_2^{\bar{\theta},c}, \hat{I}_3^{\bar{\theta},c})$ , where  $\bar{\theta}$  is the principal connection associated with the Levi-Civita connection. Let  $X$  be the vector field on  $M$  which generates the  $U(1)$ -action on  $M$  by quaternionic affine transformations defined by  $e^{i\theta} \cdot [z_0, \dots, z_n] := [e^{i\theta} z_0, \dots, e^{i\theta} z_n]$  for  $e^{i\theta} \in U(1)$  and  $[z_0, \dots, z_n] \in M$ . It holds that the Ricci tensor  $Ric^\nabla$  is  $Q$ -hermitian and  $Ric^\nabla(X, X) > 0$ . Then we can apply Corollary 6.7. Note that the action induces the well-known hyper-Kähler moment map on  $\hat{M}$  when  $c = 1$ . The hyper-Kähler metric  $G_\alpha^1 = G_\beta^1 = G_\gamma^1$  is given by a constant multiple of the standard Euclidean flat metric on  $\mathbb{H}^{n+1} \setminus \{0\}$ . Applying the  $QK/HK$ -correspondence (which amounts to taking the hyper-Kähler quotient of  $\hat{M}$  with respect to the vector field  $\hat{X}$ ) to this example yields Calabi's hyper-Kähler structure on  $T^*\mathbb{C}P^n$ .

*Hypercomplex manifold with the Obata connection:* Let  $(M, (I_1, I_2, I_3))$  be a hypercomplex manifold and  $\nabla$  its Obata connection on  $M$ . We recall the Obata connection is a canonical torsion-free connection preserving the hypercomplex structure [18]. In particular, it is a quaternionic connection with respect to the quaternionic structure  $Q = \langle I_1, I_2, I_3 \rangle$ . Assume that a vector field  $X$  with the flow  $\{\varphi_t\}_{t \in \mathbb{R}}$  on  $M$  is given, which generates a free action of  $U(1) = \mathbb{R}/2\pi\mathbb{Z}$  on  $M$  such that

$$(7.1) \quad L_X I_1 = 0, \quad L_X I_2 = 2\varepsilon I_3 \quad \text{and} \quad L_X I_3 = -2\varepsilon I_2.$$

Then it holds

$$\varphi_{-t}^* I_1 = I_1, \quad \varphi_{-t}^* I_2 = (\cos(2\varepsilon t))I_2 + (\sin(2\varepsilon t))I_3, \quad \varphi_{-t}^* I_3 = (-\sin(2\varepsilon t))I_2 + (\cos(2\varepsilon t))I_3.$$

This shows that  $X$  is a quaternionic vector field for the quaternionic structure  $Q = \langle I_1, I_2, I_3 \rangle$ . Since  $(\varphi_{-t}^* \nabla)$  is the Obata connection for the hypercomplex structure  $(\varphi_{-t}^* I_1, \varphi_{-t}^* I_2, \varphi_{-t}^* I_3)$ ,  $(\varphi_{-t}^* \nabla)$  is again a quaternionic connection for  $Q$ . By the explicit expression of the Obata connection in [5], we have

$$\frac{d}{dt}(\varphi_{-t}^* \nabla) = 0,$$

and hence  $\varphi_{-t}^* \nabla = \nabla$  for all  $t$ . It follows that  $L_X \nabla = 0$ . Because the Ricci curvature of the Obata connection is skew symmetric and  $Q$ -hermitian by Corollary 1.6 in [5], we can apply the Q/H-correspondence to  $(M, Q, \nabla)$  obtaining a hypercomplex manifold  $M'$ . The manifolds  $M$  and  $M'$  are related as follows.

**Proposition 7.2.**  *$M$  is a double covering space of  $M'$ .*

Proof. The hypercomplex structure is a global section  $s : M \rightarrow S$

$$x \mapsto s(x) = (I_1(x), I_2(x), I_3(x)),$$

and defines a global trivialization of the principal  $SO(3)$ -bundle  $S$ . Take a  $U(1)$ -invariant volume form. Since

$$\begin{aligned} \hat{M} &= \{(x, s(x)g, r) \mid x \in M, g \in SO(3), r > 0\} \\ &\cong M \times SO(3) \times \mathbb{R}^{>0} \supset M \times SO(3) \times \{1\} \cong M \times SO(3) = S \end{aligned}$$

and  $(I_1, \varphi_{-t}^* I_2, \varphi_{-t}^* I_3) = (I_1, I_2, I_3)g_{\varepsilon t} = (I_1, I_2, I_3)g_t^\varepsilon$ , where

$$(7.2) \quad g_t = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(2t) & -\sin(2t) \\ 0 & \sin(2t) & \cos(2t) \end{pmatrix},$$

we can write  $\hat{\varphi}_t(x, (I_1, I_2, I_3), r) = (\varphi_t(x), (I_1, I_2, I_3)g_{\varepsilon t}, r)$  and hence  $\hat{X}_{s(x)} = X_{s(x)}^h + (\widetilde{e_1})_{s(x)}$ . Therefore we see that  $\hat{X}_{s(x)g^\varepsilon} = X_x^h + (\widetilde{Ad_{g^{-\varepsilon}} e_1})_{s(x)g^\varepsilon}$ , where  $g \in SO(3)$ . Then the moment map  $\mu^c : \hat{M} \rightarrow \mathfrak{so}(3)(= \mathbb{R}^3)$  on  $\hat{M}$  is given by

$$\mu^c(p) = Ar^{\frac{2}{c}} \theta(\hat{X}_p) = Ar^{\frac{2}{c}} g^{-\varepsilon} e_1 g^\varepsilon$$

at any point  $p = (x, s(x)g^\varepsilon, r) \in \hat{M}$ . The level set  $P := (\mu^c)^{-1}(e_1) = (\mu^c)^{-1}((1, 0, 0))$  is given by

$$\{(x, s(x)g^\varepsilon, r) \in \hat{M} \mid x \in M, Ar^{\frac{2}{c}}g^{-\varepsilon}e_1g^\varepsilon = e_1\}.$$

Hence we have

$$(7.3) \quad P = \{(x, s(x)g^\varepsilon, A^{-\frac{c}{2}}) \in \hat{M} \mid x \in M, g \in U(1)\} \cong M \times U(1).$$

We obtain a hypercomplex manifold  $M' = P/\langle \hat{X} \rangle$ , where  $\langle \hat{X} \rangle \cong U(1)$ . Define a map  $k : M \rightarrow M'$  by  $k(x) = \pi_P((x, s(x), A^{-\frac{c}{2}}))$  for each  $x \in M$ , where  $\pi_P : P \rightarrow M' = P/\langle \hat{X} \rangle$  is the quotient map. Since  $k^{-1}(y) = \pi_P^{-1}(y) \cap (s(M) \times \{A^{-\frac{c}{2}}\})$  consists of exactly two points for each  $y \in M'$  by (7.2),  $k$  is a double covering map. By (5.7) in Lemma 5.6, it holds  $V_p = \{v \in T_p P \mid (d\mu_\alpha^c \circ \tilde{I}_\alpha^{0,c})(v) = 0\} = \mathcal{H}_p$ . It follows that  $\pi_P^*(TM') \cong \mathcal{H}|_P$ , where  $\mathcal{H}$  is the horizontal subbundle with respect to the Obata connection. Since  $s_*(Y) = Y^h$  for  $Y \in TM$ , we have

$$k_*(I_\alpha Y) = \pi_{P*}(s_* I_\alpha Y) = \pi_{P*}((I_\alpha Y)^h) = I'_\alpha(\pi_{P*}(Y^h)) = I'_\alpha(\pi_{P*} s_* Y) = I'_\alpha(k_* Y).$$

Therefore  $k : M \rightarrow M'$  is a double covering map satisfying  $k_* \circ I_\alpha = I'_\alpha \circ k_*$ .  $\square$

Note that  $M'$  is obtained by the twist data  $(X, F = 0, a = 1)$ .

**EXAMPLE 7.3.** For the Swann bundle  $\hat{M}$  of a quaternionic manifold  $(M, Q)$ , we see that  $Z_1 = -\tilde{I}_1^{0,c} Z_0^c$  satisfies the conditions required above by Lemma 3.4. So  $\hat{M}$  is a double covering space of  $(\hat{M})'$ .

*Quaternionic Hopf manifold:* Consider  $\mathbb{H}^n \cong \mathbb{R}^{4n}$  as a right-vector space over the quaternions. Set  $\tilde{M} := \mathbb{H}^n \setminus \{0\}$ . The standard hypercomplex structure  $\tilde{H} = (\tilde{I}_1, \tilde{I}_2, \tilde{I}_3)$  on  $\tilde{M}$  is defined by  $\tilde{I}_1 = R_i, \tilde{I}_2 = R_j, \tilde{I}_3 = -R_k$ , where  $R_q$  is the right-multiplication by  $q \in \mathbb{H}$ . The hypercomplex structure  $\tilde{H}$  gives a global section  $s : \tilde{M} \rightarrow S (\cong \tilde{M} \times \text{SO}(3))$  as in the previous example. The corresponding quaternionic structure is denoted by  $\tilde{Q} = \langle \tilde{I}_1, \tilde{I}_2, \tilde{I}_3 \rangle$ . Let  $\tilde{g}$  be the standard flat hyper-Kähler metric on  $\tilde{M}$ ,  $\mathcal{A} \in \text{Sp}(n)\text{Sp}(1)$  and  $\lambda > 1$ . Then  $\gamma := \lambda \mathcal{A}$  generates a group  $\Gamma = \langle \gamma \rangle$  of homotheties which acts freely and properly discontinuously on the simply connected manifold  $(\tilde{M}, \tilde{g})$ . We can identify  $\tilde{M}$  with  $\mathbb{R} \times S^{4n-1}$  by means of the diffeomorphism  $v \mapsto (t, v/\|v\|)$ , where  $t = \log \|v\| / \log \lambda$ . Under this identification,  $\gamma$  corresponds to the transformation

$$(7.4) \quad T_{\mathcal{A}} : \mathbb{R} \times S^{4n-1} \rightarrow \mathbb{R} \times S^{4n-1}, \quad (t, v) \mapsto (t + 1, \mathcal{A}v).$$

The quotient  $\tilde{M}/\Gamma \cong (\mathbb{R} \times S^{4n-1})/\langle T_{\mathcal{A}} \rangle$  is diffeomorphic to  $S^1 \times S^{4n-1}$  and inherits a quaternionic structure  $Q$  and a quaternionic connection  $\nabla$ , both invariant under the centralizer  $G^Q := Z_{\text{GL}(n, \mathbb{H})\text{Sp}(1)}(\gamma)$  of  $\gamma$  in  $\text{GL}(n, \mathbb{H})\text{Sp}(1)$ . (In particular, if  $\mathcal{A} \in \text{Sp}(n)$ , then  $\tilde{M}/\Gamma$  inherits a hypercomplex structure  $H$  and its Obata connection  $\nabla$ , both invariant under the centralizer  $G^H := Z_{\text{GL}(n, \mathbb{H})}(\gamma)$  of  $\gamma$  in  $\text{GL}(n, \mathbb{H})$ .) In fact, the quaternionic structure  $\tilde{Q}$  on  $\tilde{M}$  is  $\text{GL}(n, \mathbb{H})\text{Sp}(1)$ -invariant and induces therefore an almost quaternionic structure  $Q$  on  $\tilde{M}/\Gamma$ , since  $\Gamma \subset \text{GL}(n, \mathbb{H})\text{Sp}(1)$ . Moreover, the Levi-Civita connection  $\tilde{\nabla}$  on  $(\tilde{M}, \tilde{g})$ , which coincides with the Obata connection with respect to  $\tilde{H}$ , is invariant under all homotheties of  $\tilde{M}$ . Since  $\Gamma$  acts by homotheties, we see that  $\tilde{\nabla}$  induces a torsion-free connection  $\nabla$  on  $\tilde{M}/\Gamma$ , which preserves  $Q$ . This means that  $Q$  is a quaternionic structure on  $\tilde{M}/\Gamma$ . The group  $G^Q$  acts on  $\tilde{M}/\Gamma$  preserving the data  $(Q, \nabla)$ . If  $\mathcal{A} \in \text{Sp}(n)$ , then  $\Gamma$  preserves the hypercomplex



structure  $\tilde{H}$  on  $\tilde{M}$  and thus induces a hypercomplex structure  $H$  and  $\tilde{\nabla}$  induces the Obata connection  $\nabla$  on  $(\tilde{M}/\Gamma, H)$ . The centralizer  $G^H$  of  $\gamma = \lambda\mathcal{A}$  in  $\mathrm{GL}(n, \mathbb{H})$  acts on  $\tilde{M}/\Gamma$  preserving  $(H, \nabla)$ . We say that  $(\tilde{M}/\Gamma, Q)$  (resp.  $(\tilde{M}/\Gamma, H)$ ) is a *quaternionic* (resp. *hypercomplex*) *Hopf manifold*. Note that the hypercomplex Hopf manifolds are sometimes called quaternionic Hopf manifolds (see [19] for example).

Now taking  $\mathcal{A} = R_q$  for some unit quaternion  $q \neq \pm 1$ , we have a quaternionic Hopf manifold  $M = \tilde{M}/\Gamma$ . Then we see  $G^Q = \mathrm{GL}(n, \mathbb{H})\mathrm{U}(1) = \mathbb{R}^{>0} \times \mathrm{SL}(n, \mathbb{H})\mathrm{U}(1)$ , where  $\mathrm{U}(1)$  denotes the centralizer of  $q$  in  $\mathrm{Sp}(1)$ . Up to an automorphism of  $\mathrm{Sp}(1)$ , we can assume that

$$\mathrm{U}(1) = \{e^{i\theta} \mid \theta \in \mathbb{R}\}.$$

We take a subgroup  $\mathbb{R}^{>0} \times \mathrm{Sp}(n)\mathrm{U}(1)$  of  $G^Q$ , which acts on  $M$  transitively. The isotropy subgroup is given by  $\langle \lambda \rangle \times \mathrm{Sp}(n-1)\Delta_{\mathrm{U}(1)}$ , where  $\Delta_{\mathrm{U}(1)}$  is a diagonally embedded subgroup of  $\mathrm{Sp}(n)\mathrm{U}(1) \subset \mathrm{Sp}(n)\mathrm{Sp}(1)$  which is isomorphic to  $\mathrm{U}(1)$ . This has an expression as

$$\mathrm{Sp}(n-1)\Delta_{\mathrm{U}(1)} = \left\{ \left[ \begin{pmatrix} e^{i\theta} & 0 & \cdots & 0 \\ 0 & & & \\ \vdots & & A & \\ 0 & & & \end{pmatrix}, e^{i\theta} \right] \mid A \in \mathrm{Sp}(n-1), e^{i\theta} \in \mathrm{U}(1) \right\}.$$

As described above, we obtain an invariant quaternionic structure on the homogeneous space

$$M = (\mathbb{R}^{>0}/\langle \lambda \rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\Delta_{\mathrm{U}(1)}}.$$

REMARK 7.4. In particular, for  $n = 1$ , this yields a left invariant quaternionic structure on  $\mathrm{U}(1) \times \mathrm{Sp}(1)$ . For  $n = 2$ , we obtain an invariant quaternionic structure on the homogeneous space

$$\mathrm{U}(1) \times \frac{\mathrm{Sp}(2)\mathrm{U}(1)}{\mathrm{Sp}(1)\Delta_{\mathrm{U}(1)}} = \frac{T^2 \cdot \mathrm{Sp}(2)}{\mathrm{U}(2)}.$$

Note that the homogeneous quaternionic space  $T^2 \cdot \mathrm{Sp}(2)/\mathrm{U}(2)$  has a finite covering of the form  $(T^2 \times G)/\mathrm{U}(2)$ , where  $G$  is a compact semisimple Lie group, namely  $\mathrm{Sp}(2)$ . This presentation is of the form  $(T^k \times G)/\mathrm{U}(2)$  as considered in [15].

Consider the  $\mathrm{U}(1)$ -action on  $\tilde{M}$  defined by the right-multiplication by elements of  $\mathrm{U}(1) (\subset \mathbb{R}^{>0} \times \mathrm{Sp}(n)\mathrm{U}(1) \subset G^Q) : z \mapsto z \cdot e^{\varepsilon i t}$  ( $z \in \tilde{M}$ ). Then the corresponding vector field  $\tilde{X}$  satisfies  $\tilde{X}_z = \varepsilon z i = \varepsilon \tilde{I}_1 z$  for  $z \in \tilde{M}$ . Moreover we see that the relations (7.1) in the previous example hold, that is,  $L_{\tilde{X}} \tilde{I}_1 = 0$ ,  $L_{\tilde{X}} \tilde{I}_2 = 2\varepsilon \tilde{I}_3$ ,  $L_{\tilde{X}} \tilde{I}_3 = -2\varepsilon \tilde{I}_2$ . The  $\mathrm{U}(1)$ -action preserving the quaternionic structure induces one on  $M$  and  $\tilde{X}$  induces the vector field  $X$  on  $M$  generating the latter  $\mathrm{U}(1)$ -action on  $M$ . Considering the hypercomplex moment map on the Swann bundle  $\hat{\tilde{M}}$  (resp.  $\hat{M}$ ) of  $\tilde{M}$  (resp.  $M$ ) and the level set  $\tilde{P} \subset \hat{\tilde{M}}$  (resp.  $P \subset \hat{M}$ ) of the corresponding moment map, we can obtain a hypercomplex manifold  $\tilde{M}'$  (resp.  $M'$ ). In fact, since  $\mathrm{Ric}^{\tilde{\nabla}} = 0$  (resp.  $\mathrm{Ric}^{\nabla} = 0$ ) and  $\hat{\tilde{X}}$  (resp.  $\hat{X}$ ) is not horizontal, the  $Q/H$ -correspondence can be applied to  $\tilde{M}$  (resp.  $M$ ), cf (6.2).

Now we consider  $\tilde{M}_+ := \tilde{M}/\{\pm 1\}$  and  $M_+ := M/\{\pm 1\}$ . The quotient maps by the action of the group  $\{\pm 1\} \cong \mathbb{Z}_2$  on the manifolds are denoted by  $\tilde{\pi}_+ : \tilde{M} \rightarrow \tilde{M}_+$  and  $\pi_+ : M \rightarrow M_+$ , respectively. The induced objects on  $\tilde{M}_+$  and  $M_+$  are denoted by the same letter. We obtain

a hypercomplex isomorphism between  $\tilde{M}'$  and  $\tilde{M}_+$  as follows. Define  $\tilde{f} : \tilde{M}' \rightarrow \tilde{M}_+$  by  $\tilde{f}(x) = \tilde{\pi}_+(\hat{\pi}(u))$  for any  $x \in \tilde{M}'$ , where  $\hat{\pi} : \tilde{M} \rightarrow \tilde{M}$  is the bundle projection and  $u \in \pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{\epsilon}{2}}\})$ . Since  $\pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{\epsilon}{2}}\})$  consists of exactly two points of the form  $\{(\pm p, \tilde{H}, A^{-\frac{\epsilon}{2}})\}$  as we observed in the proof of Proposition 7.2,  $\tilde{f}$  is well-defined. It is easy to see

$$(7.5) \quad \tilde{f} \circ \pi_{\tilde{P}} = \tilde{\pi}_+ \circ \hat{\pi}$$

on  $s(\tilde{M}) \times \{A^{-\frac{\epsilon}{2}}\}$  by the definition of  $\tilde{f}$ . Furthermore we have

$$(7.6) \quad \tilde{f} \circ \tilde{k} = \tilde{\pi}_+,$$

where  $\tilde{k} : \tilde{M} \rightarrow \tilde{M}'$  is the double covering as in Proposition 7.2 for  $\tilde{M}$ . In fact, from (7.5), it follows that  $\tilde{f}(\tilde{k}(x)) = \tilde{f}(\pi_{\tilde{P}}(s(x), A^{-\frac{\epsilon}{2}})) = \tilde{\pi}_+(\hat{\pi}(s(x), A^{-\frac{\epsilon}{2}})) = \tilde{\pi}_+(x)$  for all  $x \in \tilde{M}$ .

**Lemma 7.5.** *The map  $\tilde{f} : \tilde{M}' \rightarrow \tilde{M}_+$  is an isomorphism of hypercomplex manifolds.*

*Proof.* To prove that  $\tilde{f}$  is injective, let  $x_1, x_2 \in \tilde{M}'$  such that  $\tilde{f}(x_1) = \tilde{f}(x_2)$ . There exists  $y_a \in \tilde{M}$  such that  $x_a = \pi_{\tilde{P}}(s(y_a), A^{-\frac{\epsilon}{2}})$  ( $a = 1, 2$ ). Since  $\tilde{f}(x_1) = \tilde{f}(x_2)$  and (7.5), we have  $\hat{\pi}(s(y_1), A^{-\frac{\epsilon}{2}}) = \pm \hat{\pi}(s(y_2), A^{-\frac{\epsilon}{2}})$ , that is,  $y_1 = \pm y_2$ , or equivalently  $y_1 = \varphi_0(y_2)$  or  $y_1 = \varphi_\pi(y_2)$ . Therefore, we see that  $(s(y_1), A^{-\frac{\epsilon}{2}}) = (s(\varphi_\delta(y_2)), A^{-\frac{\epsilon}{2}})$ , where  $\delta = 0$  or  $\pi$ . Then we have  $x_1 = \pi_{\tilde{P}}(s(y_1), A^{-\frac{\epsilon}{2}}) = \pi_{\tilde{P}}(s(y_2), A^{-\frac{\epsilon}{2}}) = x_2$ , which means  $\tilde{f}$  is injective. To show that  $\tilde{f}$  is surjective, let  $z \in \tilde{M}_+$  and choose  $y \in \tilde{M}$  such that  $z = \tilde{\pi}_+(y)$ . By (7.6), we obtain  $z = \tilde{\pi}_+(y) = \tilde{f}(\tilde{k}(y))$ . Hence  $\tilde{f}$  is surjective. The lift of  $v \in T\tilde{M}'$  to  $\mathcal{H}$  is denoted by  $v^{h'}$ . By

$$(I'_\alpha)_x(v) = \pi_{\tilde{P}*u}((\hat{I}_\alpha^{\tilde{H},c})_u(v^{h'})) = \pi_{\tilde{P}*u}(((I_\alpha)_{\tilde{f}(x)}(\hat{\pi}_{*u}(v^{h'})))^h),$$

then

$$\begin{aligned} \tilde{f}_{*x}((I'_\alpha)_x(v)) &\stackrel{(7.5)}{=} \tilde{\pi}_{*+}(\hat{\pi}_{*u}(((I_\alpha)_{\tilde{f}(x)}(\hat{\pi}_{*u}(v^{h'})))^h)) \\ &= (I_\alpha)_{\tilde{f}(x)}(\tilde{\pi}_{*+}(\hat{\pi}_{*u}(v^{h'}))) \stackrel{(7.5)}{=} (I_\alpha)_{\tilde{f}(x)}(\tilde{f}_{*x}(v)) \end{aligned}$$

at each point  $x \in \tilde{M}'$ , where  $u \in \pi_{\tilde{P}}^{-1}(x) \cap (s(\tilde{M}) \times \{A^{-\frac{\epsilon}{2}}\})$ . This shows that the hypercomplex manifolds  $\tilde{M}'$  and  $\tilde{M}$  are isomorphic.  $\square$

Set  $F := \tilde{f} \circ \pi_{\tilde{P}} : \tilde{P} \rightarrow \tilde{M}_+$ . Hereafter we will denote the equivalence class with respect to the action of a group  $K$  by  $[\cdot]_K$ .

**Lemma 7.6.** *We have  $F(\gamma \cdot y) = \lambda \cdot F(y)$  for all  $y \in \tilde{P}$ .*

*Proof.* For any point  $y = (p, \tilde{H}g, A^{-\frac{\epsilon}{2}})$  ( $p \in \tilde{M}$  and  $g \in \text{U}(1)$ ) of  $\tilde{P}$ , we have  $\gamma \cdot y = (\lambda p i, \tilde{H}g\rho(i), A^{-\frac{\epsilon}{2}})$  by (7.3), where  $\rho : \text{Sp}(1) \rightarrow \text{SO}(3)$  is the standard double covering. Therefore we obtain

$$\gamma \cdot [y]_{\langle \hat{X} \rangle} = [(\lambda p i, \tilde{H}g\rho(i), A^{-\frac{\epsilon}{2}})]_{\langle \hat{X} \rangle} = [(\pm \lambda p \hat{g}^{-1}, \tilde{H}, A^{-\frac{\epsilon}{2}})]_{\langle \hat{X} \rangle},$$

where  $\hat{g} \in \text{Sp}(1)$  such that  $\rho(\hat{g}) = g$ . Then it holds

$$F(\gamma \cdot y) = \tilde{f}(\gamma \cdot [y]_{\langle \hat{X} \rangle}) = \tilde{f}([( \pm \lambda p \hat{g}^{-1}, \tilde{H}, A^{-\frac{\epsilon}{2}} )]_{\langle \hat{X} \rangle}) = \tilde{\pi}_+(\pm \lambda p \hat{g}^{-1}) = \lambda \tilde{f}([y]_{\langle \hat{X} \rangle}) = \lambda F(y)$$

from (7.5).  $\square$



Note that  $M'$  has an induced  $\{\pm 1\}$ -action, since the lifted action of  $\{\pm 1\}$  to the Swann bundle  $\hat{\tilde{M}}$  commutes with  $\Gamma$  and  $U(1)$ . Let  $\pi'_+ : M' \rightarrow M'_+$  be the quotient map of the action by  $\{\pm 1\}$  on  $M'$ . We can define a map

$$\Phi : M'_+ (= \pi'_+(M')) \rightarrow \tilde{M}_+ / \langle \lambda \rangle$$

as follows. Take any  $x \in M'_+$ . Then there exists  $y \in \tilde{P}$  such that  $x = \pi'_+(\pi_P([y]_\Gamma))$  and we set  $\Phi(x) := [F(y)]_{\langle \lambda \rangle}$ . We shall show that  $[F(y)]_{\langle \lambda \rangle}$  is independent of the choice of  $y$ . If  $(x =) \pi'_+(\pi_P([y_1]_\Gamma)) = \pi'_+(\pi_P([y_2]_\Gamma))$ , there exist  $\delta \in \{\pm 1\}$ ,  $g \in U(1)$  and  $l \in \mathbb{Z}$  such that  $y_1 = \delta \cdot g \cdot \gamma^l \cdot y_2$ . By Lemma 7.6 and the definitions of  $\tilde{f}$  and  $\pi_{\tilde{P}}$ , we see

$$F(y_1) = F(\delta \cdot g \cdot \gamma^l \cdot y_2) = \lambda^l F(y_2),$$

which implies  $[F(y_1)]_{\langle \lambda \rangle} = [F(y_2)]_{\langle \lambda \rangle}$ . Moreover we have

**Lemma 7.7.** *The map  $\Phi : M'_+ \rightarrow \tilde{M}_+ / \langle \lambda \rangle$  is an isomorphism.*

*Proof.* To prove that  $\Phi$  is injective, let  $x_1, x_2 \in M'_+ = \pi'_+(M')$  such that  $\Phi(x_1) = \Phi(x_2)$ . There exists  $y_1, y_2 \in \tilde{P}$  such that  $x_a = \pi'_+(\pi_P([y_a]_\Gamma))$  ( $a = 1, 2$ ). Since  $[F(y_1)]_{\langle \lambda \rangle} = [F(y_2)]_{\langle \lambda \rangle}$ , there exists  $l \in \mathbb{Z}$  such that  $F(y_1) = \lambda^l \cdot F(y_2) = F(\gamma^l \cdot y_2)$ . Then we have  $\tilde{f}(\pi_{\tilde{P}}(y_1)) = \tilde{f}(\pi_{\tilde{P}}(\gamma^l \cdot y_2))$ , so there exists  $g \in U(1)$  such that  $y_1 = g \cdot \gamma^l \cdot y_2$ . Therefore we obtain

$$x_1 = \pi'_+(\pi_P([y_1]_\Gamma)) = \pi'_+(\pi_P([g \cdot \gamma^l \cdot y_2]_\Gamma)) = \pi'_+(\pi_P([y_2]_\Gamma)) = x_2.$$

So  $\Phi$  is injective. Next we shall show that  $\Phi$  is surjective. Take any  $z \in \tilde{M}_+ / \langle \lambda \rangle$ . There exists  $y \in \tilde{P}$  such that  $z = [F(y)]_{\langle \lambda \rangle}$ . Setting  $x = \pi'_+(\pi_P([y]_\Gamma))$ , we have  $\Phi(x) = z$ , which means  $\Phi$  is surjective. Since the hypercomplex structures are invariant under actions of all groups  $U(1) = \langle \hat{X} \rangle = \langle \hat{X} \rangle$ ,  $\Gamma$ ,  $\langle \lambda \rangle$  and  $\{\pm 1\}$  in the argument,  $\Phi$  is a hypercomplex isomorphism.  $\square$

$$\begin{array}{ccccccc}
 & & \tilde{P} & \xrightarrow{\quad / \Gamma \quad} & P & & \\
 & \swarrow \pi_{\tilde{P}} & \uparrow s \times A^{-c/2} & & \searrow \pi_P & & \\
 \tilde{M}' & \xleftarrow[\text{(as in Lemma 7.5)}]{\tilde{k}} & \tilde{M} & \xrightarrow{\quad / \Gamma \quad} & M = \tilde{M} / \Gamma & \cdots \cdots \cdots \rightarrow & M' \cong \tilde{M} / \langle \lambda \rangle \\
 & \searrow \tilde{f} & \downarrow \tilde{\pi}_+ & & \downarrow \pi_+ & & \downarrow \pi'_+ \\
 & & \tilde{M}_+ & \xrightarrow{\quad / \Gamma \quad} & M_+ = \tilde{M}_+ / \Gamma & \cdots \cdots \cdots \rightarrow & M'_+ = \pi'_+(M') \\
 & & & \searrow / \langle \lambda \rangle & & \nearrow \Phi \cong & \\
 & & & & \tilde{M}_+ / \langle \lambda \rangle & & 
 \end{array}$$

Therefore, by Lemma 7.7, the hypercomplex manifold  $M'_+$  obtained from  $M_+$  by the Q/H-correspondence is identified with  $\tilde{M}_+ / \langle \lambda \rangle$ . Since  $M'$  is the double covering space of  $M'_+$ , we have

$$M' \cong \tilde{M} / \langle \lambda \rangle.$$

The centralizer  $G^H$  of  $\lambda$  is  $\mathrm{GL}(n, \mathbb{H}) = \mathbb{R}^{>0} \times \mathrm{SL}(n, \mathbb{H})$  and take a subgroup  $\mathbb{R}^{>0} \times \mathrm{Sp}(n)$  of  $G^H$ . As we explained,  $\tilde{M}/\langle\lambda\rangle$  can be expressed by the homogeneous space

$$\tilde{M}/\langle\lambda\rangle = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}.$$

Finally, we summarize the discussion as follows.

EXAMPLE 7.8. The hypercomplex manifold

$$M' = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)}{\mathrm{Sp}(n-1)}$$

is obtained by the Q/H-correspondence from the quaternionic manifold

$$M = (\mathbb{R}^{>0}/\langle\lambda\rangle) \times \frac{\mathrm{Sp}(n)\mathrm{U}(1)}{\mathrm{Sp}(n-1)\Delta_{\mathrm{U}(1)}}.$$

(Note that we are considering the invariant quaternionic (resp. hypercomplex) structure on  $M$  (resp.  $M'$ ) described above.)

We remark that  $M'$  does not admit any hyper-Kähler structure for topological reasons, since  $M'$  is diffeomorphic to  $S^1 \times S^{4n-1}$ . Therefore our Q/H-correspondence yields examples which can not appear in the QK/HK-correspondence.

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Vicente Cortés  
 Department of Mathematics  
 and Center for Mathematical Physics  
 University of Hamburg  
 Bundesstraße 55  
 D–20146 Hamburg  
 Germany  
 e-mail: vicente.cortes@uni-hamburg.de

Kazuyuki Hasegawa  
 Faculty of teacher education  
 Institute of human and social sciences  
 Kanazawa university  
 Kakuma-machi, Kanazawa  
 Ishikawa, 920–1192  
 Japan  
 e-mail: kazuhase@staff.kanazawa-u.ac.jp